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Shell theory and its specialized branches ☆

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Abstract

For small-strain unrestricted deformation of thin elastic shells the field equations and variational principles are rederived in terms of variables immediately representing physical quantities. The relevant strain and stress tensors turn out to be identical to those commonly known as the 'modified' and the 'best'. The nonlinear theory exhibits a static-geometric duality. For orthogonal coordinates the tensor-form theory leads to a modification of the Luré–Novozhilov formulation. The general theory is specialized to that of 'quasi-shallow shells', to the membrane theory and to flexible-shell theory, which are explored with respect to basic hypotheses and accuracy.

1. Introduction

Thin-shell theory in **intrinsic form**, one avoiding any use of displacements as unknowns in the equations, is explored in Sections 2–8.

The intrinsic approach was initiated by Reissner (1912). This work introduced for the axisymmetric case those features, based on the duality of equilibrium and compatibility, which still are the effective instruments of the shell theory. The general intrinsic formulation, including the invariant-form compatibility equations, is due to Luré (1940) (although it is commonly credited to the 1941–1944 works of Synge and Chien). This break-through became possible when Goldenveizer invented in 1939 the general compatibility equations. (Cf Weatherburn, 1925 and the comprehensive review by Calladine, 1988). The nonlinear compatibility equations were derived in 1953 by Galimov and corrected by Koiter (1966). (In the applications context, a similar correction was specified by Axelrad, 1967.)

The intrinsic formulation has been further shaped by Reissner (cf Selected Works, 1996), Novozhilov (cf references in his 1970 monograph), Koiter (1960, 1980), Sanders (1963), and Budiansky and Sanders (1963). Advantages of the intrinsic approach have been ascertained by Koiter and Simmonds (1973). The intrinsic mixed formulation has proved itself numerically robust (Simmonds, 1997), unsurpassed in effectivity.

☆ Dedicated to the memory of Eric Reissner.

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A *remark* is due on the displacement approach. In the last decades it has spawned dozens of publications (reviewed by Ibrahimbegovic, 1997) and is usually justified by referring to three-dimensional elasticity. However, in the three-dimensional theory, large displacements have to be computed from the strains which are obtained by intrinsic analysis (Luré, 1970, p. 87, referring to Zak). Moreover, in certain situations, exemplified by Simmonds (1984), the displacement approach is ill-conditioned even in the linear solutions. The inexpedience of the nonlinear strain–displacement relations makes a comparison with the intrinsic alternative, at least a convincing example, indispensable.

In what follows, the **shell theory formulation** originates in the, not quite conventional, treatment of the local deformation (Sections 2 and 3). The metric and the curvature are, respectively, determined by (quasi)-vectors $\boldsymbol{a}_\alpha, \boldsymbol{b}_\alpha$ and corresponding dyadics. Subtracting the initial-geometry vectors $\boldsymbol{a}_\alpha, \boldsymbol{b}_\alpha$, rotated with the tangent plane during the deformation, from these vectors of deformed shape, yields the strain and curvature-change with the components $E_{\alpha\beta}, \rho_{\alpha\beta}$. The essential point is thereby the decomposition of the variables of *both* the rotated initial and the deformed local shape, in one and the *same* basis. The 'rotated', local reference basis, which moves with the tangent plane but does not deform, is employed. (Such basis is due to Alumae, 1956 and, in a different context, to Simmonds and Danielson, 1970.)

The so-defined strain $E_{\alpha\beta}$ is equal to the conventional $(a^*_{\alpha\beta} - a_{\alpha\beta})/2$. On the contrary, the curvature change $\rho_{\alpha\beta}$ is not identical to the standard $b^*_{\alpha\beta} - b_{\alpha\beta}$. The symmetric part of the bending-strain tensor $\rho_{\alpha\beta}$ is nothing else but the 'best modified' tensor of changes of curvature. The $\rho_{\alpha\beta}$ resolves the 'difficulty in defining a finite bending strain tensor, because the coefficient of' the moment resultant in the virtual-work expression 'is not the exact variation of anything' (Sanders, 1963). This coefficient turns out to be identical to $\delta\rho_{\alpha\beta}$, and the variations $\delta E_{\alpha\beta}$ and $\delta\rho_{\alpha\beta}$ are the virtual-work conjugates of the actual stress resultants $n^{\alpha\beta}$ and $m^{\alpha\beta}$.

The *nonlinear* equations of compatibility and equilibrium (Sections 4 and 5) display a *duality*. The principle of virtual strain and the duality render in Section 7 the principle of virtual stress and the expressions of variations of stress resultants in terms of vector stress functions (relations due to Luré (1940), although often attributed to the 1961 work of Guenther).

Any two of the three statements in Section 7—the expressions of stress resultants in terms of stress functions, the compatibility equations for the increments of strain and the principle of virtual stress—confirm the third one (as do the dual three statements in Section 6).

The elastic energy and the equations of elasticity, just as the strain and equilibrium, are expressed in terms of physically defined strain and stress resultants (Section 8). No 'modified' variables are required.

The field equations are presented also in orthogonal coordinates.

The **specialized branches** of shell theory are treated in Sections 9–12 which include the relevant literary notes. But are the specialized branches still required? Do not computers and numerics make the general theory sufficient for all problems? They do not—the contemporary judgments of leaders in shell-structures analysis (whether in Moscow or in Virginia, U.S.A.) can be approximated by the 18th century sentence: "One must have hypotheses and theories to organize his results, else all remains sheer garbage" (Lichtenberg).

The *axisymmetric* theory is not the theme of what follows: it is not a simplified branch, but an integral part of the general theory. However, a remark on the axisymmetric analysis is due, owing to its role in the nonlinear problems and as an interface to the flexible-shell theory. The nonlinear axisymmetric equations, given in 1949–1950 by Reissner 1996, (pp. 225ff, 275ff), have been extended to the Saint-Venant bending problem and are instrumental in the treatment of 'long' tubes and open profiles (Axelrad, 1960, 1962, 1987). Emmerling (1982) proved the approximate treatment of 1962 to be reasonably adequate. (The problem was reconsidered in 1981 by Boyle, Int. J. Solids Structures, p. 515. If allowance is made for the use of $E/(1-\nu^2)$ instead of E (cf Section 11), Boyle's results confirm once more those of 1962, provided the 1981 solution is original, as assured on its pp. 515, 525 and 527: "As far as the author is aware the only other study of the problem is ... by Kostovetskii").

The work of the early 1960s made it clear that the Saint-Venant approach to thin-walled tubes and open-sections does not have real promise for applications—the resistance to bending was found to be reduced drastically by curvature. This has shown it to be as drastically fortified by stiffening ribs or debilitated by buckling. These factors, excluded in the Saint-Venant solutions, had to be considered. The instruments have been provided by flexible-shell theory and, respectively, by local-stability analysis (Axelrad, 1965). These ushered in the two-dimensional treatment for the problems of Karman and Brazier, for large non-axisymmetric displacements of shells of revolution, and for the stability analysis of states shaped by two-dimensionally variable deformation.

The **specialized branches** listed in the summary follow from the general theory on the basis of specific hypotheses. The error of these assumptions turns out to depend on: (i) the variation of the stress state, measured by intervals of variation L_α and (ii) the variation of the unit normal vector $\boldsymbol{n}$, measured by principal curvatures $1/R_\alpha$. For *quasi-shallow* shell theory (Section 9) the error estimate is L_α^2/R_α^2; for the *membrane* theory (Section 10) it is Rh/L^2 ($R = \min|R_\alpha|$, $L = \min L_\alpha$). Thus, the first theory turns out to be adequate to the general theory for stress states which vary with *both* surface coordinates much more intensely than the unit normal vector $\boldsymbol{n}$. The membrane theory is adequate for states varying with *both* coordinates less intensely than $\boldsymbol{n}$. The third theory (Section 11) describes shells *designed for* large *deformability* by small strain. These, flexible, shells have stress states which vary less intensely with *one* surface coordinate than with the other. The flexible-shell theory covers the domain between those of the other two specialized theories. The three simplified theories together are adequate for nearly all realistic shell states (Section 12, Fig. 1).

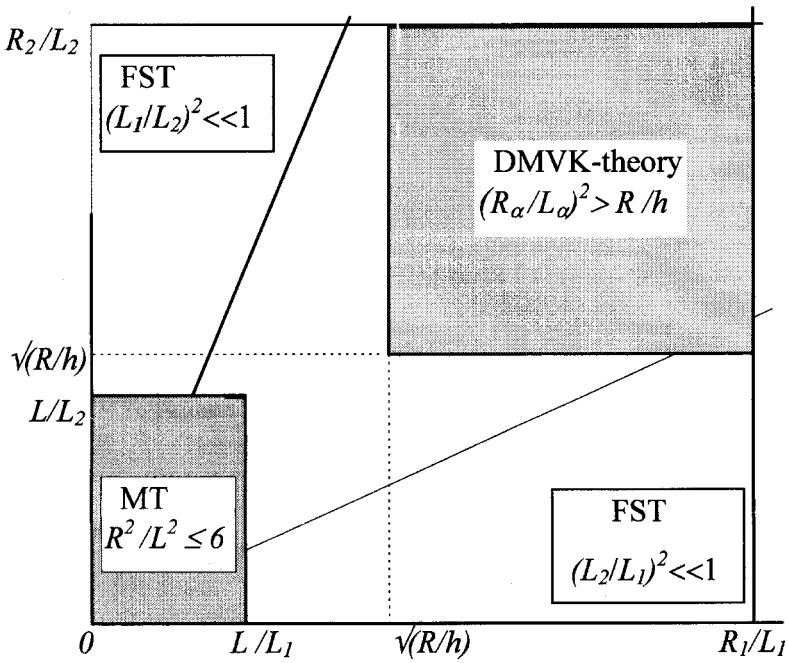

Fig. 1. Together the membrane theory (MT), the DMVK-theory and the flexible-shell theory (FST) are adequate for virtually any realistic shell stress state.

2. Shell shape, vector description

The reference surface is fixed at the middle of the wall thickness. This is the optimum choice for homogeneous isotropic shells (cf, e.g., Axelrad, 1987). The radius vector $\boldsymbol{r}$, from a fixed pole to any point of the reference surface, is determined by Gaussian coordinates x^α (the Greek-letter indices take the values 1, 2). The coordinate $x^3 \equiv z$ is defined as the distance of a point of shell volume to the reference surface, the z-lines are straight and normal to this surface.

The tangential base vectors $\boldsymbol{a}_\alpha, \boldsymbol{a}^\beta$ and the unit normal vector of the surface $\boldsymbol{a}_3 \equiv \boldsymbol{n}$ are defined and denoted by

$$\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha} = a_{\alpha\beta}\boldsymbol{a}^\beta, \quad a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad \boldsymbol{a}_\alpha \cdot \boldsymbol{a}^\alpha = 1, \quad \boldsymbol{a}_1 \cdot \boldsymbol{a}^2 = \boldsymbol{a}^1 \cdot \boldsymbol{a}_2 = 0,$$

$$a_\alpha \equiv |\boldsymbol{a}_\alpha| = \sqrt{a_{\alpha\alpha}}, \quad a^\alpha \equiv |\boldsymbol{a}^\alpha| = \sqrt{a^{\alpha\alpha}}, \quad a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta;$$

$$\boldsymbol{n} = \boldsymbol{a}_1 \times \boldsymbol{a}_2/\sqrt{a}, \quad a = |\boldsymbol{a}_1 \times \boldsymbol{a}_2|^2, \quad a_1/a^2 = a_2/a^1 = \sqrt{a}. \tag{2.1}$$

The comma preceding a subscript α denotes partial differentiation with respect to x^α.

The derivatives of the unit normal vector $\boldsymbol{n}$ indicate the curvature of the surface, they constitute curvature vectors $\boldsymbol{b}_\alpha$ (Axelrad and Emmerling, 1988):

$$\boldsymbol{b}_\alpha \equiv b_{\alpha\beta}\boldsymbol{a}^\beta = \boldsymbol{n}_{,\alpha}, \quad b_{\alpha\beta} = \boldsymbol{n}_{,\alpha} \cdot \boldsymbol{a}_\beta, \quad b_\alpha^\beta \equiv \boldsymbol{b}_\alpha \cdot \boldsymbol{a}^\beta = b_{\alpha\lambda}a^{\lambda\beta}. \tag{2.2}$$

Here and in the following the Einstein summation convention is employed. The $b_{\alpha\beta}$ defined above is equal to that of Sanders (1963). The more widely used definition amounting to $b_{\alpha\beta} = -\boldsymbol{n}_{,\alpha} \cdot \boldsymbol{a}_\beta$ would cause additional minus signs in several formulas, e.g., in (2.3), (2.4) and (3.1).

The two fundamental tensors of the surface—the metric tensor $\boldsymbol{a}^\alpha\boldsymbol{a}_\alpha$ and the curvature tensor $\boldsymbol{a}^\alpha\boldsymbol{b}_\alpha$—can, following Simmonds (1997), be defined in the coordinate-free form—with the aid of the *del* operator—as $\nabla\boldsymbol{r}$ and $\nabla\boldsymbol{n}$.

The local shape of the reference surface, and of the x^α lines on it, will be described in terms of the rotation of a plane tangent to the surface, when this plane is shifted along the coordinate lines. The position of the tangent plane is fixed by the normal vector $\boldsymbol{n}(x^\alpha)$. This leads to the definition (2.2) of $\boldsymbol{b}_\alpha$. To determine a position of the plane also in its movement around the normal, the tangent plane is considered to be bound to a linear element of the surface identified by its directional unit vector $\boldsymbol{t}$. Thus, the local shape of the surface is determined by curvature vectors $\boldsymbol{k}_\alpha$ which, unlike the $\boldsymbol{b}_\alpha$, include $\boldsymbol{n}$-components. The $\boldsymbol{k}_\alpha$ determines the derivatives of $\boldsymbol{n}, \boldsymbol{t}$, and of any other unit vector $\boldsymbol{v}$ bound to the tangent plane:

$$\boldsymbol{k}_\alpha = \boldsymbol{n} \times \boldsymbol{b}_\alpha + l_\alpha\boldsymbol{n}, \quad [\boldsymbol{n} \quad \boldsymbol{t} \quad \boldsymbol{v}]_{,\alpha} = \boldsymbol{k}_\alpha[\boldsymbol{n} \quad \boldsymbol{t} \quad \boldsymbol{v}]. \tag{2.3}$$

The curvature vector $\boldsymbol{k}_\alpha$ represents the angle $\boldsymbol{k}_\alpha \mathrm{d}x^\alpha$ between the planes tangent to the surface at two points with a distance $\boldsymbol{a}_\alpha \mathrm{d}x^\alpha$ between them.

(The term $\boldsymbol{n}l_\alpha$ in the $\boldsymbol{k}_\alpha$ determines the in-plane curvature of the coordinate line x^α, when the angle between this line and the $\boldsymbol{t}$ is constant along the middle surface, which is the reasonable choice of $\boldsymbol{t}$ for orthogonal x^α.)

The three coordinates x^α, z label a material point; they are not changed by deformation. The other variables defined above take at a point of the deformed middle surface new values which will be denoted by an asterisk superscript—$\boldsymbol{n}^*, \boldsymbol{b}_\alpha^*, \boldsymbol{l}_\alpha^*, \boldsymbol{a}_\alpha^* \ldots$. With the new values of variables replacing the initial ones, the relations (2.3) determine $\boldsymbol{k}_\alpha^*$ and the derivatives of $\boldsymbol{n}^*, \boldsymbol{t}^*, \boldsymbol{v}^*$. For a continuous surface the radius

vector $\boldsymbol{r}^*$ and any unit vector $\boldsymbol{v}^*$ bound to the tangent plane are continuous functions of x^α—for any current shape (including the undeformed one with $\boldsymbol{r}, \boldsymbol{v}$) the continuity conditions:

$$\boldsymbol{v}^*_{,12} - \boldsymbol{v}^*_{,21} = 0, \quad \boldsymbol{r}^*_{,12} = \boldsymbol{r}^*_{,21} \tag{2.4}$$

are fulfilled. These conditions lead to equations for the $\boldsymbol{k}^*_\alpha$ and $\boldsymbol{a}^*_\alpha$. Applying to $\boldsymbol{v}^*_{,1}$ and $\boldsymbol{v}^*_{,2}$ the derivation formulas (2.3) and using the relation $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) - \boldsymbol{B}\times(\boldsymbol{A}\times\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}$, makes the left side of the first of eqns (2.4) equal to a product of a certain expression and $\boldsymbol{v}^*$. As $\boldsymbol{v}^*$ is arbitrary, its cofactor must be zero. This renders an equation which, together with the equation following from (2.4) and (2.1), constitute the continuity conditions of the deformed surface

$$\boldsymbol{k}^*_{1,2} - \boldsymbol{k}^*_{2,1} + \boldsymbol{k}^*_1 \times \boldsymbol{k}^*_2 = 0, \quad \boldsymbol{a}^*_{1,2} = \boldsymbol{a}^*_{2,1}. \tag{2.5}$$

These relations are, of course, valid also for the initial shape of the surface—for the $\boldsymbol{k}_\alpha$ and $\boldsymbol{a}_\alpha$. The first of eqns (2.5) is equivalent to the three scalar relations of Gauss and Codazzi.

3. The surface deformation

A neighbourhood of a point of a deforming surface will be regarded in two stages: first, after its rigid-body movement, second, after the entire deformation. This description will use an auxiliary, local reference basis $\boldsymbol{a}^0_\alpha$, which is not deformed (Alumae, 1955; Simmonds and Danielson, 1970). The vectors $\boldsymbol{a}^0_\alpha$ indicate positions attained by the undeformable basis $\boldsymbol{a}_\alpha$ after it has moved together with the tangent plane: $\boldsymbol{a}^0_\alpha \cdot \boldsymbol{a}^0_\alpha \equiv \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\alpha\beta}$.

(If at some point of the surface the tangent plane does not move with respect to the observer, then at this point $\boldsymbol{a}^0_\alpha = \boldsymbol{a}_\alpha$. Furthermore, the undeformable basis $\boldsymbol{a}^0_\alpha$ can be made identical to $\boldsymbol{a}_\alpha$. This is the case when $\boldsymbol{a}_\alpha$ are defined to be fixed to the tangent plane and to move with the plane to its current position. Of course, after the x^α-lines have been deformed, these $\boldsymbol{a}_\alpha$ are not anymore tangent to the x^α-lines, as are $\boldsymbol{a}^*_\alpha = \boldsymbol{r}^*_{,\alpha}$. By such definition of $\boldsymbol{a}_\alpha$, the superscript '0' becomes unnecessary: the rotated local reference basis $\boldsymbol{a}_\alpha$ can hardly be confused with the $\boldsymbol{a}_\alpha$ of the undeformed shape—the basis $\boldsymbol{a}_\alpha$ need not appear in the analysis simultaneously in its initial position, when $\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}$, and in the rotated position.)

A unit normal vector $\boldsymbol{a}_3 = \boldsymbol{a}^3 = \boldsymbol{n}$ moves into the unit normal of the deformed surface $\boldsymbol{a}^0_3 = \boldsymbol{a}^{3_0} = \boldsymbol{n}^*$. (The strain in the volume will be considered in Section 8, on the basis of the Kirchhoff hypotheses.)

Any vector $\boldsymbol{V}(x^\alpha) = V_j\boldsymbol{a}^j$ moving together with the tangent plane of the reference surface into its current state rotates into $\boldsymbol{V}^0(x^\alpha) = V_j\boldsymbol{a}^{j_0}$—the components of $\boldsymbol{V}^0$ with respect to the rotated basis $\boldsymbol{a}^{j_0}$ are equal to those of $\boldsymbol{V}$ with respect to the initial basis $\boldsymbol{a}^j$. Specifically, the vectors $\boldsymbol{a}_\alpha, \boldsymbol{b}_\alpha$ and $\boldsymbol{k}_\alpha$, bound to the tangent plane and rotated without being deformed, become $\boldsymbol{a}^0_\alpha, \boldsymbol{b}^0_\alpha$ and $\boldsymbol{k}^0_\alpha$, determined by the decompositions:

$$\boldsymbol{a}^0_\alpha = a_{\alpha\beta}\boldsymbol{a}^{\beta_0}, \quad \boldsymbol{b}^0_\alpha = b_{\alpha\beta}\boldsymbol{a}^{\beta_0}, \quad \boldsymbol{k}^0_\alpha = \boldsymbol{n}^* \times \boldsymbol{b}^0_\alpha + l_\alpha \boldsymbol{n}^*. \tag{3.1}$$

The strain of the reference surface changes $\boldsymbol{a}^0_\alpha, \boldsymbol{b}^0_\alpha$ and $\boldsymbol{k}^0_\alpha$ into the current-state vectors $\boldsymbol{a}^*_\alpha, \boldsymbol{b}^*_\alpha$ and $\boldsymbol{k}^*_\alpha$. All variables describing the current, deformed state will be decomposed in the local reference basis: $\boldsymbol{a}^0_\alpha, \boldsymbol{a}^{\beta_0}, \boldsymbol{a}^0_3 \equiv \boldsymbol{a}^{3_0} \equiv \boldsymbol{n}^*$. The corresponding components will be denoted by a prime superscript:

$$\boldsymbol{a}^*_\alpha \equiv a^*_{\alpha\beta}\boldsymbol{a}^{\beta*} \equiv a'_{\alpha\beta}\boldsymbol{a}^{\beta_0}; \quad \boldsymbol{b}^*_\alpha \equiv b^*_{\alpha\beta}\boldsymbol{a}^{\beta*} \equiv b'_{\alpha\beta}\boldsymbol{a}^{\beta_0}, \quad b^\beta_\alpha = b'_{\alpha\lambda}a^{\lambda\beta}. \tag{3.2}$$

An advantage of the basis $\boldsymbol{a}^0_\alpha$, displayed in (3.2): the raising and lowering of indices of tensors referred to this basis is done with the metric of the initial geometry.

The local deformation of the surface will be described by strain vectors denoted $\boldsymbol{E}_\alpha$, $\boldsymbol{\rho}_\alpha$ and $\boldsymbol{K}_\alpha$. The

definition of these vectors is the crucial step. The $\boldsymbol{E}_\alpha$, $\boldsymbol{\rho}_\alpha$ and $\boldsymbol{K}_\alpha$ are defined and measured as the difference between the vectors $\boldsymbol{a}^*_\alpha, \boldsymbol{b}^*_\alpha$ and $\boldsymbol{k}^*_\alpha$ of the current shape, and the respective rotated vectors $\boldsymbol{a}^0_\alpha, \boldsymbol{b}^0_\alpha$ and $\boldsymbol{k}^0_\alpha$ (which are the $\boldsymbol{a}_\alpha, \boldsymbol{b}_\alpha$ and $\boldsymbol{k}_\alpha$, moved with the tangent plane, but not otherwise changed). With (3.1) and (3.2) the two strain vectors and their components $E_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are defined by the following relations, in which it is essential that all the vectors, $\boldsymbol{a}^*_\alpha$ and $\boldsymbol{a}^0_\alpha, \boldsymbol{b}^*_\alpha$ and $\boldsymbol{b}^0_\alpha$, are decomposed in the same basis:

$$\boldsymbol{E}_\alpha \equiv E_{\alpha\beta}\boldsymbol{a}^{\beta_0} = \boldsymbol{a}^*_\alpha - \boldsymbol{a}^0_\alpha, \quad E_{\alpha\beta} = a'_{\alpha\beta} - a_{\alpha\beta};$$

$$\rho_\alpha \equiv \rho_{\alpha\beta}\boldsymbol{a}^{\beta_0} = \boldsymbol{b}^*_\alpha - \boldsymbol{b}^0_\alpha, \quad \rho_{\alpha\beta} = b'_{\alpha\beta} - b_{\alpha\beta}. \tag{3.3}$$

The components $E_{\alpha\beta}$ of the strain vector $\boldsymbol{E}_\alpha$ represent the extensional strain and—through the angles between $\boldsymbol{a}^*_\alpha$ and $\boldsymbol{a}^0_\alpha$—the shear strain. This is displayed clearer by the physical components, in Section 4. The physical sense of the curvature-change $\rho_\alpha = \boldsymbol{b}^*_\alpha - \boldsymbol{b}^0_\alpha$ becomes more graphic by recalling that $\boldsymbol{b}^*_\alpha = \boldsymbol{n}^*_{,\alpha}$ and $\boldsymbol{b}^0_\alpha = (\boldsymbol{n}_{,\alpha})^0$.

The other curvature-change vector $\boldsymbol{K}_\alpha$ is defined and determined as the change of the $\boldsymbol{k}^0_\alpha$ to $\boldsymbol{k}^*_\alpha$:

$$\boldsymbol{K}_\alpha = \boldsymbol{k}^*_\alpha - \boldsymbol{k}^0_\alpha = \boldsymbol{n}^* \times \boldsymbol{\rho}_\alpha + \lambda_\alpha \boldsymbol{n}^*, \quad \lambda_\alpha = l^*_\alpha - l_\alpha. \tag{3.4}$$

The strain measures defined by (3.3) and (3.4) (Axelrad and Emmerling, 1988) originate from the vectorial strain–displacement relations of Reissner's 1974 work (Reissner, 1996, p. 353ff).

The $\boldsymbol{E}_\alpha$ and $\boldsymbol{\rho}_\alpha$ can, following Simmonds (1997), be defined in the coordinate-free form using the del operator ∇ stated in terms of the rotated basis—$\nabla(\,) = \boldsymbol{a}^{\alpha_0}(\,)_{,\alpha}$.

The choice of the basis $\boldsymbol{a}^0_\alpha$, that is of its position with respect to $\boldsymbol{a}^*_\alpha$, influences the definition of the $\boldsymbol{E}_\alpha, \boldsymbol{\rho}_\alpha$—the relations (3.3). Different positions of the $\boldsymbol{a}^0_\alpha$ mean different partitions of the shear angle into the two angles between $\boldsymbol{a}^*_\alpha$ and $\boldsymbol{a}^0_\alpha$. Specifically, the tensor $\boldsymbol{a}^0_\alpha \boldsymbol{E}_\alpha$ can be made symmetric (Alumae, 1955). The corresponding condition $E_{\alpha\beta} = E_{\beta\alpha}$ takes, with (3.3) the form $\boldsymbol{a}^*_\alpha \cdot \boldsymbol{a}^0_\beta = \boldsymbol{a}^*_\beta \cdot \boldsymbol{a}^0_\alpha$, or:

$$a'_{\alpha\beta} = a'_{\beta\alpha} \quad (E_{\alpha\beta} = E_{\beta\alpha}). \tag{3.5}$$

The standard tensors $(a^*_{\alpha\beta} - a_{\alpha\beta})/2$ and $b^*_{\alpha\beta} - b_{\alpha\beta}$ are simply expressed in terms of $E_{\alpha\beta}$ and $\rho_{\alpha\beta}$. The first quantitatively coincides with the $E_{\alpha\beta}$. However, the $b^*_{\alpha\beta} - b_{\alpha\beta}$ (in the literature, 'an obvious choice for the strain measures') is different from $\rho_{\alpha\beta}$. Indeed, with (3.2), (3.3) and (3.5):

$$a^*_{\alpha\beta} - a_{\alpha\beta} = \boldsymbol{a}^*_\alpha \cdot \boldsymbol{a}^*_\beta - a_{\alpha\beta} = a'_{\alpha\beta} + E_{\beta\alpha} - a_{\alpha\beta} = 2E_{\alpha\beta}; \tag{3.6}$$

$$b^*_{\alpha\beta} - b_{\alpha\beta} = \boldsymbol{b}^*_\alpha \cdot \boldsymbol{a}^*_\beta - b_{\alpha\beta} = \rho_{\alpha\beta} + \boldsymbol{b}^*_\alpha \cdot \boldsymbol{E}_\beta, \quad \boldsymbol{b}^*_\alpha \cdot \boldsymbol{E}_\beta = b'_{\beta\lambda} E^\lambda_\alpha. \tag{3.7}$$

4. Compatibility equations

The strain vectors $\boldsymbol{E}_\alpha, \boldsymbol{K}_\alpha$ must satisfy two vector compatibility equations following from (2.5) with $\boldsymbol{a}^*_\alpha = \boldsymbol{a}^0_\alpha + \boldsymbol{E}_\alpha$ and $\boldsymbol{k}^*_\alpha = \boldsymbol{k}^0_\alpha + \boldsymbol{K}_\alpha$, according to (3.3) and (3.4).

The intended transformation of (2.5) requires derivatives of the rotated vectors. The $\boldsymbol{n}(x^\alpha + \mathrm{d}x^\alpha)$ constitutes with $\boldsymbol{n}(x^\alpha)$ the angle $\boldsymbol{k}_\alpha\,\mathrm{d}x^\alpha$. The angle between the rotated vectors $\boldsymbol{n}^*(x^\alpha + \mathrm{d}x^\alpha)$ and $\boldsymbol{n}^*(x^\alpha)$ is $\boldsymbol{k}^*_\alpha \mathrm{d}x^\alpha$. With $\boldsymbol{k}^*_\alpha = \boldsymbol{k}^0_\alpha + \boldsymbol{K}_\alpha$ from (3.4) this gives $\boldsymbol{a}^0_{\alpha,\beta}$ and a similar formula for any vector $\boldsymbol{V}^0$ defined as a vector $\boldsymbol{V}$ rotated with the tangent plane:

$$\boldsymbol{a}^0_{\alpha,\,\beta} = (\boldsymbol{a}_{\alpha,\,\beta})^0 + \boldsymbol{K}_\beta \times \boldsymbol{a}^0_\alpha, \quad \boldsymbol{V}^0_{,\beta} = \left(\boldsymbol{V}_{,\beta}\right)^0 + \boldsymbol{K}_\beta \times \boldsymbol{V}^0. \tag{4.1}$$

Thus, the derivatives of rotated vectors are equal to the rotated derivatives of the vectors in the initial state plus a term reflecting the bending strain. The formula (4.1) is useful also for determining the covariant derivatives.

Insert into (2.5) the expressions for $\boldsymbol{a}^*_\alpha, \boldsymbol{k}^*_\alpha$ and their derivatives, from (3.3), (3.4), (4.1), and take into account, the eqns (2.5) for the initial shape—for $\boldsymbol{a}_\alpha$ and $\boldsymbol{k}_\alpha$. This leads to equations:

$$\boldsymbol{K}_{1,2} - \boldsymbol{K}_{2,1} + \boldsymbol{K}_1 \times \boldsymbol{K}_2 + \boldsymbol{q}_c = 0, \quad \left(\boldsymbol{E}_{\alpha,\beta} + \boldsymbol{K}_\beta \times \boldsymbol{a}^0_\alpha\right)\varepsilon^{\alpha\beta} + \boldsymbol{m}_c/\sqrt{a} = 0. \tag{4.2}$$

Here $\varepsilon^{\alpha\beta}$ is the Levi–Civita permutation tensor: $\varepsilon^{\alpha\alpha} = 0$, $\varepsilon^{12} = -\varepsilon^{21} = 1/\sqrt{a}$.

The nonlinear vector compatibility equations (4.2) (Axelrad, 1981) are an extension of the linear equation of Reissner (1974; 1996, p. 353). In the recent work (Simmonds, 1997), these equations have become a nearly self-evident consequence of the new, hybrid form of the theory.

The 'load' terms $\boldsymbol{q}_c$ and $\boldsymbol{m}_c$ serve in (4.2) to complement the analogy with the equations of equilibrium, they may represent temperature expansions or be equal to zero.

For a small enough increment of deformation, the nonlinear term $\boldsymbol{K}_1 \times \boldsymbol{K}_2$ of eqns (4.2) is negligible. With the small increments of $\boldsymbol{K}_\alpha, \boldsymbol{E}_\alpha$ and of $\boldsymbol{q}_c$, $\boldsymbol{m}_c$ denoted by $\mathbf{K}_\alpha, \mathbf{E}_\alpha$ and $\mathbf{q}_c$, $\boldsymbol{m}_c$ (different font), the eqns (4.2) can be written for the increments as:

$$\mathbf{K}_{1,2} - \mathbf{K}_{2,1} + \mathbf{q}_c = 0, \quad \left(\mathbf{E}_{\alpha,\beta} + \mathbf{K}_\beta \times \boldsymbol{a}^0_\alpha\right)\varepsilon^{\alpha\beta} + \mathbf{m}_c/\sqrt{a} = 0. \tag{4.3}$$

In these equations, the nonlinearity is represented only implicitly—as seen from their derivation, the eqns (4.3), just as (4.2), are referred to the current shape. The derivatives take it into account as exemplified in (4.1). This nonlinearity becomes explicit in the component form of the eqns (4.2), which is obtained in the standard way, with the decompositions of $\boldsymbol{E}_\alpha$ and $\boldsymbol{K}_\alpha$ defined by (3.1), (3.3) and (3.4). The corresponding six equations of compatibility for $\boldsymbol{q}_c$, $\boldsymbol{m}_c = 0$ (e.g., those of Axelrad and Emmerling, 1988, without terms of relative magnitude of the strain) contain nonlinear terms with factors $b^{\delta'}_\alpha \lambda_\beta$, $\rho_{\alpha\lambda}\rho_{\beta\sigma}$ and $E^\lambda_\alpha b'_{\beta\lambda}$:

$$\varepsilon^{\alpha\beta}\left(\varepsilon^{\delta\lambda}\rho_{\alpha\lambda;\beta} + b^{\delta'}_\alpha \lambda_\beta\right) = 0, \tag{4.4}$$

$$\varepsilon^{\alpha\beta}\lambda_{\alpha;\beta} + \varepsilon^{\alpha\beta}\varepsilon^{\sigma\lambda}\rho_{\alpha\lambda}\left(b_{\beta\sigma} + \rho_{\beta\sigma}/2\right) = 0, \tag{4.5}$$

$$\lambda_\beta = -\varepsilon^{\sigma\kappa}E_{\sigma\beta;\kappa}, \tag{4.6}$$

$$\varepsilon^{\alpha\beta}\left(\rho_{\beta\alpha} + E^\lambda_\alpha b'_{\beta\lambda}\right) = 0. \tag{4.7}$$

The semicolon subscript denotes the covariant derivative, which in all cases concerns components with respect to the 'rotated' local reference basis $\boldsymbol{a}^0_\alpha$ and $\boldsymbol{a}^{\beta_0}$. The derivatives of this basis are determined by the formulas (4.1). For small strain the $b^{\delta'}_\alpha$ and $b'_{\beta\lambda}$ can be replaced in (4.4) and (4.7) by the initial curvature components b^δ_α and $b_{\beta\lambda}$.

The expression in the brackets in (4.7) coincides with $b^*_{\alpha\beta} - b_{\alpha\beta}$ [cf (3.7)]. This, and the symmetry of $b^*_{\alpha\beta} - b_{\alpha\beta}$, does not, however, constitute a sufficient reason for employing the $b^*_{\alpha\beta} - b_{\alpha\beta}$ as the main dependent variable. It can be noted, further, that the $\boldsymbol{n}^*$-projection of the second eqn (4.2), which constitutes the eqn (4.7), involves the ρ_{12}, ρ_{21}, but not $\rho_{\alpha\alpha}$.

The eqn (4.7) determines the skew part of the bending–strain tensor $\rho_{\alpha\beta}$ and, thus, renders its remaining, symmetric part $\bar{\rho}_{\alpha\beta}$. With the simplifications $b'_{\beta\lambda}E^\lambda_\beta \cong b_{\beta\lambda}E^\lambda_\beta$ this gives:

$$\rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} + \left(b_{\alpha\lambda}E^{\lambda}_{\beta} - b_{\beta\lambda}E^{\lambda}_{\alpha}\right), \quad \bar{\rho}_{\alpha\beta} = \left(\rho_{\alpha\beta} + \rho_{\beta\alpha}\right)/2. \tag{4.8}$$

Consider now some of the relations treated above in terms of *orthogonal coordinates* x^{α} and physical components, which are defined and denoted by:

$$a_{12} = 0, \quad \sqrt{a} = a_1 a_2, \quad \boldsymbol{t}_{\alpha} \equiv \boldsymbol{t}^{\alpha} = \boldsymbol{a}^0_{\alpha}/a_{\alpha}, \quad \boldsymbol{a}^{\alpha_0} = \boldsymbol{a}^0_{\alpha}/a_{\alpha\alpha}, \quad a^{\alpha} = 1/a_{\alpha}, \tag{4.9}$$

$$\boldsymbol{E}_1/a_1 = \boldsymbol{t}_1\varepsilon_1 + \boldsymbol{t}_2\gamma/2, \quad \varepsilon_1 = E_{11}/a_1a_1, \quad \gamma/2 = E_{12}/a_1a_2 \ (1\ 2), \tag{4.10}$$

$$\boldsymbol{k}^*_{\alpha}/a_{\alpha} = \boldsymbol{n}^* \times \boldsymbol{t}^{\beta}/R'_{\alpha\beta} + \boldsymbol{n}^*\kappa_{\alpha 3}, \quad \kappa_{\alpha 3} = \lambda_{\alpha}/a_{\alpha}, \quad 1/R'_{\alpha\beta} = b'_{\alpha\beta}/a_{\alpha}a_{\beta},$$

$$\kappa_{\alpha\beta} \equiv \rho_{\alpha\beta}/a_{\alpha}a_{\beta}, \quad 1/R'_{\alpha j} = 1/R_{\alpha j} + \kappa_{\alpha j}, \quad 1/R_{\alpha 3} = -\varepsilon^{\alpha\beta}a_{\alpha,\beta}. \tag{4.11}$$

The sign (1 2), after a relation with specified indices 1 and 2, implies another relation, obtainable when the indices are replaced, respectively, by 2 and 1.

The variables ε_{α} and γ denote the relative extension and the shear angle.

The curvature radii $R_{\alpha\alpha}$ of the normal sections, and the physical components $\kappa_{\alpha\beta}$ of the bending and torsional strain, will also be denoted in the form usual in the technical literature:

$$R_{\alpha} = R_{\alpha\alpha}, \quad R'_{\alpha} = R'_{\alpha\alpha}, \quad \kappa_{\alpha} = \kappa_{\alpha\alpha}, \quad \tau_1 = \kappa_{12}, \quad \tau_2 = \kappa_{21}, \quad \tau = \bar{\rho}_{12}/a_1a_2. \tag{4.12}$$

The relations (4.11) and (4.12) without the indices '*' or '′' are valid for the initial shape.

For the orthogonal coordinates, the compatibility equations follow with (4.9)–(4.12) from (4.4)–(4.7) or directly from (4.2):

$$\varepsilon^{\alpha\beta}(a_{\beta}\kappa_{\beta 2})_{,\alpha} + \kappa_{13}/R'_{21} - \kappa_{23}/R_1 + \tau_2/R_{13} - \kappa_1/R'_{23} = 0; \quad (1\ 2) \tag{4.13}$$

$$\varepsilon^{\alpha\beta}(a_{\beta}\kappa_{\beta 3})_{,\alpha} + \kappa_1/R'_2 + \kappa_2/R_1 - \tau_1/R'_{21} - \tau_2/R_{12} = 0; \tag{4.14}$$

$$\kappa_{13} = (a_2\gamma)_{,1}/\left(2\sqrt{a}\right) - (a_1\varepsilon_1)_{,2}/\sqrt{a} + \gamma/2R'_{23} - \varepsilon_2/R'_{13},$$

$$\kappa_{23} = (a_2\varepsilon_2)_{,1}/\sqrt{a} - (a_1\gamma)_{,2}/\left(2\sqrt{a}\right) + \gamma/2R'_{13} - \varepsilon_1/R'_{23}; \tag{4.15}$$

$$\tau_1 = \tau + (\tau_1 - \tau_2)/2, \quad \tau_1 - \tau_2 = \gamma/2R'_2 - \gamma/2R'_1 + \varepsilon_1/R'_{21} - \varepsilon_2/R'_{12} \ (1\ 2). \tag{4.16}$$

With the accuracy of the small-strain theory, some $\varepsilon_{\alpha}, \gamma$-terms in (4.11)–(4.16) can be linearized. In particular, the expressions of $1/R'_{\alpha j}$ following from (4.11) and (4.15) indicate:

$$[\varepsilon_{\alpha} \quad \gamma]/R'_{\beta 3} \cong [\varepsilon_{\alpha} \quad \gamma]/R_{\beta 3}. \tag{4.17}$$

5. Equilibrium

Tractions acting in the normal sections of the shell are represented by resultants reduced to the reference surface. The force and moment resultants of tractions acting on the length $a_2\,\mathrm{d}x^2$ (not

encompassing the extension) of the section $x^1 = \text{const}$ are defined and determined in the forms

$$\begin{bmatrix}\boldsymbol{T}^1 & \boldsymbol{G}^1\end{bmatrix}\mathrm{d}x^2 = \begin{bmatrix}\boldsymbol{N}^1 & \boldsymbol{M}^1\end{bmatrix}a_2\,\mathrm{d}x^2 = \begin{bmatrix}\boldsymbol{n}^1 & \boldsymbol{m}^1\end{bmatrix}\sqrt{a}\,\mathrm{d}x^2 \quad (1\,2). \tag{5.1}$$

The $\boldsymbol{N}^1$ and $\boldsymbol{M}^1$ are the resultant force and moment per unit length of the x^2-line.

Consider the equilibrium of an element of the shell bounded by sections x^α, $x^\alpha + \mathrm{d}x^\alpha = \text{const}$ and encompassing the reference-surface area $\mathrm{d}A = \sqrt{a}\,\mathrm{d}x^1\,\mathrm{d}x^2$. Denote by $\boldsymbol{q}\,\mathrm{d}A, \boldsymbol{m}\,\mathrm{d}A$ the force and moment resultants of the load acting on the element $\mathrm{d}A$ of the reference surface. The equilibrium equations for the element are readily obtained in the form:

$$\boldsymbol{T}^\alpha_{,\alpha} + \boldsymbol{q}\sqrt{a} = 0, \quad \boldsymbol{G}^\alpha_{,\alpha} + \boldsymbol{a}^*_\alpha \times \boldsymbol{T}^\alpha + \boldsymbol{m}\sqrt{a} = 0. \tag{5.2}$$

Here the non-linearity resulting from the deformed shape of the shell is taken into account in the derivatives [as indicated after (4.3)]. For small strain $\boldsymbol{a}^*_\alpha$ can be replaced in (5.2) by $\boldsymbol{a}^0_\alpha$.

The components of the stress resultants and those of the distributed load are defined and denoted:

$$\boldsymbol{n}^\alpha = n^{\alpha j}\boldsymbol{a}^0_j, \quad \boldsymbol{m}^\alpha = m^{\alpha\lambda}\boldsymbol{n}^* \times \boldsymbol{a}^0_\lambda, \quad \boldsymbol{q} = q^\alpha \boldsymbol{a}^0_\alpha + q\boldsymbol{n}^* \quad (j = 1, 2, 3). \tag{5.3}$$

The component equations of equilibrium follow from (5.2) in the mathematical way (cf Luré, 1970, p. 885). These equations, nonlinear in the $b^{\beta\prime}_\alpha$, $b'_{\alpha\beta}$-terms, are (for $\boldsymbol{m} = 0$):

$$n^{\alpha\beta}_{;\alpha} + b^{\beta\prime}_\alpha n^{\alpha 3} + q^\beta = 0, \tag{5.4}$$

$$n^{\alpha 3}_{;\alpha} - b'_{\alpha\beta} n^{\alpha\beta} + q = 0, \tag{5.5}$$

$$n^{\alpha 3} = m^{\lambda\alpha}_{;\lambda}, \tag{5.6}$$

$$'n^{\alpha\beta} - {}'n^{\beta\alpha} = 0, \quad {}'n^{\alpha\beta} = n^{\alpha\beta} - b^{\beta\prime}_\lambda m^{\lambda\alpha}. \tag{5.7}$$

The eqns (5.7) determine the skew part of the tensor $n^{\alpha\beta}$ and, thus, its symmetric part $\bar{n}^{\alpha\beta}$

$$n^{\alpha\beta} = \bar{n}^{\alpha\beta} + \left(m^{\lambda\alpha} b^\beta_\lambda - m^{\lambda\beta} b^\alpha_\lambda\right)/2, \quad \bar{n}^{\alpha\beta} = \left(n^{\alpha\beta} + n^{\beta\alpha}\right)/2. \tag{5.8}$$

Here, with the accuracy of the small-strain theory we have set $b^{\alpha\prime}_\lambda = b^\alpha_\lambda$.

The duality of the vector equations of compatibility (4.2) and of equilibrium (5.2) is made transparent by their simplicity. This static-geometric analogy renders each equation of equilibrium from the dual equation of compatibility, and vice versa, by the replacement of variables:

$$\begin{aligned}&\begin{bmatrix}\boldsymbol{T}^1 & \boldsymbol{T}^2 & \boldsymbol{q}\end{bmatrix} \equiv \begin{bmatrix}\boldsymbol{n}^1\sqrt{a} & \boldsymbol{n}^2\sqrt{a} & \boldsymbol{q}\end{bmatrix} \Leftrightarrow \begin{bmatrix}\boldsymbol{K}_2 & -\boldsymbol{K}_1 & \boldsymbol{K}_1 \times \boldsymbol{K}_2 + \boldsymbol{q}_c\end{bmatrix},\\ &\begin{bmatrix}\boldsymbol{G}^1 & \boldsymbol{G}^2 & \boldsymbol{m}\end{bmatrix} \equiv \begin{bmatrix}\boldsymbol{m}^1\sqrt{a} & \boldsymbol{m}^2\sqrt{a} & \boldsymbol{m}\end{bmatrix} \Leftrightarrow [\boldsymbol{E}_2 \quad -\boldsymbol{E}_1 \quad \boldsymbol{m}_c].\end{aligned} \tag{5.9}$$

This duality is perturbed merely by the nonlinear term $\boldsymbol{K}_1 \times \boldsymbol{K}_2$ of the compatibility eqns (4.2). This term can be made negligible in the incremental, step-by-step, solutions, usual for nonlinear problems. Between the equilibrium eqns (5.2) and the compatibility eqns (4.3) for the increments there are the simpler duality relations

$$\begin{bmatrix}\boldsymbol{T}^1 & \boldsymbol{T}^2 & \boldsymbol{q}\end{bmatrix} \equiv \begin{bmatrix}\boldsymbol{n}^1\sqrt{a} & \boldsymbol{n}^2\sqrt{a} & \boldsymbol{q}\end{bmatrix} \Leftrightarrow \begin{bmatrix}\mathbf{K}_2 & -\mathbf{K}_1 & \mathbf{q}_c\end{bmatrix},$$

$$\begin{bmatrix}\boldsymbol{G}^1 & \boldsymbol{G}^2 & \boldsymbol{m}\end{bmatrix} \equiv \begin{bmatrix}\boldsymbol{m}^1\sqrt{a} & \boldsymbol{m}^2\sqrt{a} & \boldsymbol{m}\end{bmatrix} \Leftrightarrow [\mathbf{E}_2 \quad -\mathbf{E}_1 \quad \mathbf{m}_c]. \tag{5.10}$$

The static-geometric analogy (5.9) and (5.10) extends to the nonlinear theory the duality relations which the theory owes to the work of Luré (1940) (cf Goldenveizer, 1976).

For orthogonal coordinates x^α, the stress resultants (5.1) are conveniently decomposed in physical components $N^{\alpha j}$, $M_{\alpha\beta}$. These are directly expressed in terms of the $n^{\alpha j}$, $m^{\alpha\lambda}$ defined in (5.3) and presented in the simpler notation usual in the technical literature:

$$\boldsymbol{N}^\alpha = N^{\alpha j}\boldsymbol{t}_j, \quad \boldsymbol{t}_3 \equiv \boldsymbol{n}^*, \quad N^{\alpha j} = n^{\alpha j}a_\alpha a_j, \quad \boldsymbol{M}^\alpha = M_{\alpha\beta}\boldsymbol{n}^* \times \boldsymbol{t}^\beta, \quad M_{\alpha\beta} = m^{\alpha\beta}a_\alpha a_\beta;$$
$$N_1 = N^{11}, \quad S_1 = N^{12}, \quad Q_\alpha = N^{\alpha 3}, \quad S = (S_1 + S_2)/2, \quad M_1 = M^{11}, \quad H \cong M^{12} \quad (1\,2). \tag{5.11}$$

In terms of the physical components defined in (5.11), the equilibrium equations can be obtained by specialization of (5.4)–(5.7), or directly from (5.2). Another possibility—these equilibrium equations follow from the compatibility eqns (4.13)–(4.16) by replacing the variables according to the nonlinear duality rule (5.9):

$$[\kappa_\alpha \quad \tau_\alpha \quad \varepsilon_\alpha \quad \gamma/2 \quad \kappa_{13} \quad \kappa_{23}] \Leftrightarrow \begin{bmatrix}-N_\beta & S_\beta & M_\beta & H & -Q_2 & Q_1\end{bmatrix} \quad (\beta \neq \alpha); \tag{5.12}$$

$$\begin{bmatrix}R_{1j} & R'_{2j}\end{bmatrix} \Leftrightarrow \begin{bmatrix}R'_{1j} & R'_{2j}\end{bmatrix}. \tag{5.13}$$

The relations (5.13) take into account the small-strain simplifications (4.17). As dual to the terms with the factors $1/R'_{1j}$ in the equilibrium equations, the duality (5.13) indicates in the equations of compatibility terms with the initial curvatures $1/R_{1j}$. Another combination of the components of the term $\boldsymbol{K}_1 \times \boldsymbol{K}_2$ of (4.2) results in the alternative to (5.13) relations $[R'_{1j} \quad R_{2j}] \Leftrightarrow [R'_{1j} \quad R'_{2j}]$.

The balance of moments around $\boldsymbol{n}^*$—the eqn (5.7)—determines the shear resultants S_α in terms of their symmetric part $S = (S_1 + S_2)/2$:

$$S_1 = S + (H/R_1 - H/R_2 + M_1/R_{12} - M_2/R_{21})/2, \quad S = \bar{n}^{12}/a_1a_2 \quad (1\,2). \tag{5.14}$$

This definition of S differs from the Luré–Novozhilov formulation (cf Novozhilov, 1970), which employs the variable S identical to the quantity $'n^{12}/a_1a_2$ with $'n^{12}$ as defined in (5.7).

6. Principle of virtual strain

Independently of material properties of the shell and of its arbitrary previous deformation, the principle of virtual strain, equates the work done by the external forces on any virtual deformation to the corresponding work δW done by the internal forces:

$$\int\int (\boldsymbol{q}\cdot\delta\boldsymbol{u} + \boldsymbol{m}\cdot\delta\vartheta)\,\mathrm{d}A + \int (\boldsymbol{N}\cdot\delta\boldsymbol{u} + \boldsymbol{M}\cdot\delta\vartheta)\,\mathrm{d}s = \int\int \delta W\,\mathrm{d}A. \tag{6.1}$$

The integrals $\int\int(\,)\,\mathrm{d}A$ and $\int(\,)\,\mathrm{d}s$ extend over the entire middle-surface area A inside the shell, and, respectively, over the boundary contour (s) of this area; s is the length measured along the (s). The A and s are measured before the deformation. The $\delta\boldsymbol{u}$ and $\delta\vartheta$ denote the virtual displacement and rotation at a point of the middle surface; $\boldsymbol{N}\mathrm{d}s$ and $\boldsymbol{M}\mathrm{d}s$ are the force and moment resultants of the forces acting on the edge length $\mathrm{d}s$.

Consider the work δW done on a shell element bounded by surfaces x^α, $x^\alpha + \mathrm{d}x^\alpha = \text{ const}$. At the edges of the element the stress resultants $\boldsymbol{T}^\alpha\mathrm{d}x^\beta$, $\boldsymbol{G}^\alpha\mathrm{d}x^\beta$ $(\beta \neq \alpha)$ work on the virtual displacement and

rotation of an edge $x^\alpha + \mathrm{d}x^\alpha = \text{const}$ with respect to its opposite edge $x^\alpha = \text{const}$. Let $\boldsymbol{r}(x^\alpha)$ and $\boldsymbol{r}^*(x^\alpha)$ denote, respectively, the radius vector of the surface directly before and after the virtual deformation. The dimensions of the element are $\boldsymbol{r}_{,\alpha}\mathrm{d}x^\alpha$. The corresponding to $\boldsymbol{T}^\alpha \mathrm{d}x^\beta$ virtual displacement is $(\delta\boldsymbol{u})_{,\alpha}\mathrm{d}x^\alpha$ minus the contribution $\delta\vartheta \times \boldsymbol{r}_{,\alpha}\mathrm{d}x^\alpha$ of the rigid-body rotation. The virtual deformation rotates $\boldsymbol{r}_{,\alpha}$ into $\boldsymbol{a}^0_\alpha = \boldsymbol{r}_{,\alpha} + \delta\vartheta \times \boldsymbol{r}_{,\alpha}$, which makes $\delta\boldsymbol{a}^0_\alpha = \delta\vartheta \times \boldsymbol{r}_{,\alpha}$. Finally, $(\delta\boldsymbol{u})_{,1} = \delta\boldsymbol{r}^*_{,1} = \delta\boldsymbol{a}^*_1$, and, from (3.3), $\boldsymbol{a}^*_1 - \boldsymbol{a}^0_1 = E_1$. The virtual work of the resultant $\boldsymbol{T}^1\mathrm{d}x^2$ is:

$$\boldsymbol{T}^1\,\mathrm{d}x^2 \cdot \left[(\delta\boldsymbol{u})_{,1} - \delta\vartheta \times \boldsymbol{a}^0_1\right]\mathrm{d}x^1 = \boldsymbol{T}^1\,\mathrm{d}x^2 \cdot \left[\delta\boldsymbol{a}^*_1 - \delta\boldsymbol{a}^0_1\right]\mathrm{d}x^1 = \boldsymbol{T}^1\,\mathrm{d}x^2 \cdot \delta E_1\,\mathrm{d}x_1. \tag{6.2}$$

The moment resultant $\boldsymbol{G}^\alpha$ works on the virtual rotation $(\delta\vartheta)_{,\alpha}\,\mathrm{d}x^\alpha$, which is the virtual change of the angle $\boldsymbol{k}^*_\alpha\,\mathrm{d}x^\alpha$ between the edges of the element. The definition (3.4) $\boldsymbol{K}_\alpha = \boldsymbol{k}^*_\alpha - \boldsymbol{k}^0_\alpha$, leads to the expression of the angle $(\delta\vartheta)_{,\alpha}\,\mathrm{d}x^\alpha$ in terms of the curvature change $\boldsymbol{K}_\alpha$:

$$(\delta\vartheta)_{,\alpha}\,\mathrm{d}x^\alpha = \delta\boldsymbol{k}^*_\alpha\,\mathrm{d}x^\alpha = \delta\boldsymbol{K}_\alpha\,\mathrm{d}x^\alpha. \tag{6.3}$$

With (6.2) and (6.3) the work of the inner forces, accounted for in the principle (6.1), is

$$\delta W = (\boldsymbol{T}^\alpha \cdot \delta E_\alpha + \boldsymbol{G}^\alpha \cdot \delta\boldsymbol{K}_\alpha)\,\mathrm{d}x^1\,\mathrm{d}x^2/\mathrm{d}A = (\boldsymbol{n}^\alpha \cdot \delta E_\alpha + \boldsymbol{m}^\alpha \cdot \delta\boldsymbol{K}_\alpha) = n^{\alpha\beta}\delta E_{\alpha\beta} + m^{\alpha\beta}\delta\rho_{\alpha\beta}. \tag{6.4}$$

There follow also from (6.2) and (6.3) the equations between the virtual strains and virtual displacements

$$\delta E_\alpha = (\delta\boldsymbol{u})_{,\alpha} + \boldsymbol{a}^*_\alpha \times \delta\vartheta, \quad \delta\boldsymbol{K}_\alpha = (\delta\vartheta)_{,\alpha}, \tag{6.5}$$

where it is taken into account that the difference between the $\boldsymbol{a}^*_\alpha$ and $\boldsymbol{a}^0_\alpha$ is by (3.3) the infinitely small virtual strain: $\boldsymbol{a}^0_\alpha = \boldsymbol{a}^*_\alpha - \delta E_\alpha$.

The linear-theory expressions of the strains and changes of curvature follow from (6.5), when the virtual rotations and displacements are equal to small actual ones: $E_\alpha = \boldsymbol{u}_{,\alpha} + \boldsymbol{a}_\alpha \times \vartheta$, $\boldsymbol{K}_\alpha = \vartheta_{,\alpha}$.

The first of these equations renders, with the zero transverse shear $E_\alpha \cdot \boldsymbol{n} = 0$ (Section 8), the rotations in terms of displacements.

Each of the three statements—the principle (6.1), the equilibrium equations (5.2) and the eqns (6.5) between virtual strain and displacements—follows from the other two. Consider these relations which confirm (6.5), (5.2) and (6.1).

The virtual strain–displacement relations (6.5) can be obtained by inserting into the principle, (6.1) with (6.4), expressions of $\boldsymbol{q}$, $\boldsymbol{m}$ in terms of the stress resultants, which follow from the equilibrium eqns (5.2).

The equilibrium eqns (5.2) follow from the principle (6.1) and (6.4) after inserting the virtual strain–displacement relations (6.5). Besides the equilibrium equations, this yields the boundary conditions for the edge loading and also the variational, Galerkin, form of the equations and the conditions.

The equilibrium equations in terms of only the symmetric tensors $\bar{n}^{\alpha\beta}$ and $m^{\alpha\beta}$ follow from the principle when the internal work (6.4) is transformed to

$$\delta W = \bar{n}^{\alpha\beta}\delta E_{\alpha\beta} + m^{\alpha\beta}\delta\bar{\rho}_{\alpha\beta}. \tag{6.6}$$

The equality of the first term of (6.6) to that in (6.4) is assured by $E_{\alpha\beta} = E_{\beta\alpha}$ of (3.5). The second term of (6.6) is made approximately equal to $m^{\alpha\beta}\delta\rho_{\alpha\beta}$ of (6.4) by the known (e.g., Axelrad and Emmerling, 1988) relation

$$m^{\alpha\beta} = m^{\beta\alpha}\left[1 + 0(h/R)\right], \tag{6.7}$$

where R is the minimum absolute value of the normal-section curvature radius at a point of the reference surface, $0(h/R)$ has the order-of-magnitude of the shell-theory error (Section 8).

Finally, the principle of virtual strain (6.1) is confirmed by the virtual strain–displacement relations (6.5) and the equilibrium equations (5.2)—with $\boldsymbol{q}$, $\boldsymbol{m}$, $\delta\boldsymbol{u}$ and $\delta\vartheta$ expressed by (5.2) and (6.5) in terms of stress resultants and virtual strains, the left of (6.1) turns out identical to its right-hand side specified by (6.4).

The equilibrium equations in terms of the variables ${}'n^{\alpha\beta}$, defined in (5.7), follow from the principle (6.1) with the expression of δW which results from (6.4) when $\rho_{\alpha\beta}$ is presented by (3.7) in terms of the $b^*_{\alpha\beta} - b_{\alpha\beta}$

$$\delta W = {}'n^{\alpha\beta}\delta E_{\alpha\beta} + m^{\alpha\beta}\delta\left(b^*_{\alpha\beta} - b_{\alpha\beta}\right). \tag{6.8}$$

7. Principle of virtual stress

The duality relations (5.10) suggest a principle, which is entirely dual to that of virtual strain in (6.1) and (6.4). It is the principle of virtual stress governing the strain increments $\mathbf{K}$, $\mathbf{E}$, $\mathbf{K}_\alpha$ and $\mathbf{E}_\alpha$:

$$\int\int\left(\mathbf{q}_c\cdot\delta\boldsymbol{f} + \mathbf{m}_c\cdot\delta\boldsymbol{g}\right)\mathrm{d}A + \int\left(\mathbf{K}\cdot\delta\boldsymbol{f} + \mathbf{E}\cdot\delta\boldsymbol{g}\right)\mathrm{d}s = -\int\int\delta W\,\mathrm{d}A. \tag{7.1}$$

The internal work δW is given by the expression dual to (6.4). With the accuracy equivalent to that of $m^{\alpha\beta} = m^{\beta\alpha}$ in (6.7), it is determined by only the symmetric parts of increments of the relevant strains. Retaining for the increments of $\bar{\rho}_{\alpha\beta}$, $E_{\alpha\beta}$ the notation of these variables themselves we have:

$$\delta W = \mathbf{K}_\alpha\cdot\delta\boldsymbol{m}^\alpha + \mathbf{E}_\alpha\cdot\delta\boldsymbol{n}^\alpha = \bar{\rho}_{\alpha\beta}\delta m^{\alpha\beta} + E_{\alpha\beta}\delta\bar{n}^{\alpha\beta}. \tag{7.2}$$

The formulation (7.1) and (7.2) of the virtual-stress principle (Axelrad and Emmerling, 1990) avoids any use of displacements or their gradients—it is totally intrinsic.

The principle of virtual stress renders the compatibility eqns (4.3) for the increments $\mathbf{K}_\alpha, \mathbf{E}_\alpha$ of unlimited small-strain deformation, i.e. of arbitrary displacements and rotations. The sole restriction—the increment of deformation must be small enough to obviate the term $\mathbf{K}_1 \times \mathbf{K}_2$. Of course, this restriction does not hinder the use of the principle for nonlinear problems.

A further manifestation of the static-geometric duality, stated in (5.10), are relations dual to (6.5). Just as the strain resultants determined by (6.5) identically satisfy the linear compatibility equations, or those for increments, the stress resultants, dual to these strains, have the following expressions which identically satisfy the equlibrium equations (by $\boldsymbol{q}, \boldsymbol{m} = 0$)

$$\delta\boldsymbol{m}^\alpha\varepsilon_{\alpha\beta} = (\delta\boldsymbol{f})_{,\beta} + \boldsymbol{a}^*_\alpha\times\delta\boldsymbol{g}, \quad \delta\boldsymbol{n}^\alpha\varepsilon_{\alpha\beta} = (\delta\boldsymbol{g})_{,\beta}\ \left(\varepsilon_{\alpha\beta} = \varepsilon^{\alpha\beta}a\right). \tag{7.3}$$

The stress functions $\boldsymbol{f}$ and $\boldsymbol{g}$, introduced here, are dual to the $\boldsymbol{u}$ and ϑ of (6.5).

The eqns (7.3) can be traced back to those given by Luré (1940).

There are three basic statements dealt with in the current Section 7: the compatibility eqns (4.3), the formulas (7.3), which express the variations of the stress resultants in terms of the stress functions, and the principle (7.1). The three statements and the relations between them are dual to those which are basic in Section 6. Each of the three can be recovered, and thus confirmed, by means of the other two.

Specifically, the principle (7.1) renders, in conjunction with (7.3), the compatibility eqns (4.3) and the boundary conditions for strain, as well as the Galerkin form of both. The compatibility eqns (4.3) and the principle (7.1) render the eqns (7.3). Finally, the compatibility eqns (4.3) and the formulas (7.3) for

$\delta \boldsymbol{m}^{\alpha}$, $\delta \boldsymbol{n}^{\alpha}$ confirm the principle (7.1). That is, the virtual work—the left side of (7.1) with (4.3) and (7.3)—turns out identical to the expression (7.2) for δW.

8. Elasticity relations

The strain and stress in the volume of a thin shell are determined on the basis of the following (Kirchhoff–Love) hypotheses:

(a) In the analysis of strain, particles comprising a straight line normal to the middle surface can be assumed to constitute such a normal after the deformation, and the change of distance between these particles can be disregarded.
(b) In the stress–strain relations the influence of the stresses acting on the sections parallel to the plane tangent to the reference surface can be disregarded.

These assumption are known (e.g., Koiter and Simmonds, 1973) to introduce errors which have the relative order of magnitude of

$$\max\left(h/R, h^2/L^2, h^2/d^2, \eta\right) \ll 1. \tag{8.1}$$

Here η and $1/R$ are the maximum absolute values of the principal strain in the shell and of the normal-section curvature; d denotes the distance to the shell edge.

In (8.1) and in the following, L is the minimum of intervals of variation L_{α} of any function $\boldsymbol{F}(x^{\alpha})$ which is substantial in the description of the stress and deformation (cf Koiter, 1966, p. 20):

$$1/L_{\alpha} \sim |\delta \boldsymbol{F}/a_{\alpha}\delta x^{\alpha}|/F_{\mathrm{m}}, \quad F_{\mathrm{m}} = \max|\boldsymbol{F}|, \quad L = \min L_{\alpha}. \tag{8.2}$$

(The sign ' $\sim$ ' between two quantities indicates the equality of their orders of magnitude.)

The physical meaning of L_1 is clarified by a simple case $F = \sin(x^1/c)$, $a_1 = 1$. The definition (8.2) gives in this case, the interval of variation $L_1 = c$.

A direct expression of the hypothesis (a) is constituted by the following formulas for the radius vector $\boldsymbol{R}^*$ of a point in the deformed shell volume ($\boldsymbol{R}^*$ has no direct relation to the maximum normal-section curvature $1/R$) and by formulas for the deformed and the rotated bases $(\boldsymbol{g}^*_{\alpha}, \boldsymbol{g}^0_{\alpha})$ of the three-dimensional metric

$$\boldsymbol{R}^* = \boldsymbol{r}^* + z\boldsymbol{n}^*, \quad \boldsymbol{g}^*_{\alpha} \equiv \boldsymbol{R}^*_{,\alpha} = \boldsymbol{a}^*_{\alpha} + z\boldsymbol{b}^*_{\alpha}, \quad \boldsymbol{g}^0_{\alpha} = \boldsymbol{a}^0_{\alpha} + z\boldsymbol{b}^0_{\alpha}. \tag{8.3}$$

The definition (3.3) of the reference-surface strain (of $\boldsymbol{E}_{\alpha}$, $\boldsymbol{\rho}_{\alpha}$) is now to be extended—the strain in the shell volume is defined and denoted by $\boldsymbol{\gamma}_{\alpha}$ and $\gamma_{\alpha\beta}$ which are expressed in terms of $\boldsymbol{E}_{\alpha}$, $\boldsymbol{\rho}_{\alpha}$:

$$\boldsymbol{\gamma}_{\alpha} \equiv \gamma_{\alpha\beta}\boldsymbol{a}^{\beta_0} = \boldsymbol{g}^*_{\alpha} - \boldsymbol{g}^0_{\alpha} = \boldsymbol{E}_{\alpha} + z\boldsymbol{\rho}_{\alpha}, \quad \gamma_{\alpha\beta} = E_{\alpha\beta} + z\boldsymbol{\rho}_{\alpha\beta}. \tag{8.4}$$

With the basis vectors $\boldsymbol{g}^*_{\alpha}, \boldsymbol{g}^0_{\alpha}$ of (8.3), the standard Cauchy strain formula gives:

$$\begin{aligned} &\left(g^*_{\alpha\beta} - g_{\alpha\beta}\right)/2 = \gamma_{\alpha\beta} + \Delta_{\alpha\beta}, \quad g^*_{\alpha\beta} = \boldsymbol{g}^*_{\alpha}\cdot\boldsymbol{g}^*_{\beta}, g_{\alpha\beta} = \boldsymbol{g}^0_{\alpha}\cdot\boldsymbol{g}^0_{\beta}, \\ &2\Delta_{\alpha\beta} = z\left(\boldsymbol{E}_{\alpha}\cdot\boldsymbol{b}^*_{\beta} + \boldsymbol{E}_{\beta}\cdot\boldsymbol{b}^*_{\alpha}\right) + z^2\left(\boldsymbol{\rho}_{\alpha}\cdot\boldsymbol{b}^*_{\beta} + \boldsymbol{\rho}_{\beta}\cdot\boldsymbol{b}^*_{\alpha}\right) + \left(\boldsymbol{g}^*_{\alpha} - \boldsymbol{g}^0_{\alpha}\right)\cdot\left(\boldsymbol{g}^*_{\beta} - \boldsymbol{g}^0_{\beta}\right). \end{aligned} \tag{8.5}$$

The term $\Delta_{\alpha\beta}$ has the estimate:

$$\Delta_{\alpha\beta} = (h/R)0(\gamma_{\alpha\beta}). \tag{8.6}$$

This means, the term $\Delta_{\alpha\beta}$ can be neglected in (8.5)—its relative magnitude does not exceed the error (8.1) of the basic hypotheses of the thin-shell theory.

For thin shells, the vector definition (8.4) of the strain $\gamma_{\alpha\beta}$ is equivalent to the standard one

$$\gamma_{\alpha\beta} = E_{\alpha\beta} + z\rho_{\alpha\beta} \cong \left(g^*_{\alpha\beta} - g_{\alpha\beta}\right)/2.$$

With the z-term of $\Delta_{\alpha\beta}$ retained, the formula (8.5) amounts to $(g^*_{\alpha\beta} - g_{\alpha\beta})/2 = E_{\alpha\beta} + z(b^*_{\alpha\beta} - b_{\alpha\beta})$. This determines the strain not by the $\rho_{\alpha\beta}$, as in (8.4), but, with equivalent accuracy, by $b^*_{\alpha\beta} - b_{\alpha\beta}$ [given in terms of $\rho_{\alpha\beta}$ by (3.7)].

For a shell made of Hookean elastic material, the strain energy per unit area of the reference surface is in accordance with the assumption (b) determined by the formula (Koiter, 1960):

$$V = \frac{1}{2}\int \frac{E}{1+v}\left(g^{\alpha\lambda}g^{\alpha\mu} + \frac{v}{1+v}g^{\alpha\beta}g^{\lambda\mu}\right)\gamma_{\alpha\beta}\gamma_{\lambda\mu}\sqrt{g}\,\mathrm{d}z, \tag{8.7}$$

where the integral extends over the shell thickness, E denotes the modulus of elasticity and v is Poisson's ratio.

The definitions of $g_{\alpha\beta}$, $a_{\alpha\beta}$ and $g = \det|g_{\alpha\beta}|$ indicate the estimates:

$$g_{\alpha\beta} = a_{\alpha\beta} + (z/R)0(a_{\alpha\beta}), \quad g^{\alpha\beta} = a^{\alpha\beta} + (z/R)0\left(a^{\alpha\beta}\right), \quad g = a + (z/R)0(a). \tag{8.8}$$

With this, the elastic-energy formula (8.7) decomposes into a main term and an additive of relative magnitude of h/R—of the error (8.1) of the thin-shell theory. This additive has to be dispensed with. The elastic energy is determined solely by $E_{\alpha\beta}$ and $\bar{\rho}_{\alpha\beta}$—the symmetric components of the strain vector $\boldsymbol{E}_\alpha$ and of the symmetric part $\boldsymbol{a}^{\alpha_0}\boldsymbol{a}^{\beta_0}\bar{\rho}_{\alpha\beta}$ of the tensor $\boldsymbol{a}^{\alpha_0}\boldsymbol{a}^{\beta_0}\rho_{\alpha\beta}$:

$$V = \frac{1}{2}\frac{E}{1+v}\left(a^{\alpha\lambda}a^{\beta\mu} + \frac{v}{1+v}a^{\alpha\beta}a^{\lambda\mu}\right)\left(E_{\alpha\beta}E_{\lambda\mu}h + \bar{\rho}_{\alpha\beta}\bar{\rho}_{\lambda\mu}h^3/12\right). \tag{8.9}$$

A similar expression can be obtained from (8.7) for nonhomogeneous and anisotropic material. It must, however, retain a term determined by products of the membrane and bending strains. This term can be minimized by appropriate choice of the reference surface (cf, e.g., Axelrad, 1987). For the present case of homogeneous isotropic material the appropriate reference surface is the middle surface and the bounds of integration in (8.7) leading to (8.9) are $z = -h/2$ and $h/2$.

The variation of the elastic energy δV is equal to the corresponding virtual work of the inner forces (6.6):

$$\delta V = \delta W = \bar{n}^{\alpha\beta}\delta E_{\alpha\beta} + m^{\alpha\beta}\delta\bar{\rho}_{\alpha\beta}. \tag{8.10}$$

For the mutually independent virtual strains and (8.9), this gives the elasticity relations

$$\bar{n}^{\alpha\beta} = \delta V/\partial E_{\alpha\beta} = \frac{Eh}{1-v^2}\left[(1-v)E^{\alpha\beta} + va^{\alpha\beta}E^\alpha_\alpha\right],$$

$$m^{\alpha\beta} = \partial V/\partial\bar{\rho}_{\alpha\beta} = D\left[(1-v)\bar{\rho}_{\alpha\beta} + va^{\alpha\beta}\bar{\rho}^\alpha_\alpha\right], \quad D = Eh^3/\left[12(1-v^2)\right]. \tag{8.11}$$

Solved with respect to the strain resultants these relations become

$$E_{\alpha\beta} = (1/Eh)[(1+v)\bar{n}^{\alpha\beta} - va_{\alpha\beta}\bar{n}^\alpha_\alpha], \quad \bar{\rho}_{\alpha\beta} = (12/Eh^3)[(1+v)m^{\alpha\beta} - va_{\alpha\beta}m^\alpha_\alpha]. \tag{8.12}$$

The elastic energy in terms of the stress resultants is determined by inserting (8.12) into (8.9), and the constitutive eqns (8.12) directly confirm the variation of the elastic energy δV to be equal to an expression, similar to (7.2):

$$\delta V(m_{\alpha\beta}, n_{\alpha\beta}) = \delta W = \bar{\rho}_{\alpha\beta}\delta m^{\alpha\beta} + E_{\alpha\beta}\delta\bar{n}^{\alpha\beta}. \tag{8.13}$$

For orthogonal coordinates x^{α} the elastic-energy density and the constitutive eqns (8.11) and (8.12), presented in terms of the physical components of strain and stress, defined in (4.10), (4.12), (5.10) and (5.11), are

$$2V = \left(N_1^2 + N_2^2 - 2\nu N_1 N_2\right)/Eh + S^2/Gh + D\left(\kappa_1^2 + \kappa_2^2 + 2\nu\kappa_1\kappa_2\right) + \tau^2 Gh^3/3, \tag{8.14}$$

$$\varepsilon_1 Eh = N_1 - \nu N_2, \quad \gamma Gh = S, \quad M_2/D = \kappa_2 + \nu\kappa_1, \quad H/\left(Gh^3/6\right) = \tau \quad (1\ 2). \tag{8.15}$$

The energy expression (8.14) is a mixed one: the membrane part is determined in terms of the resultants N_1, N_2, and S, the bending part—in terms of the strain resultants dual, in the sense of (5.12), to N_1, N_2, and S. Between these two parts of V, as well as between the two corresponding groups of elasticity equations in (8.15) can be observed a duality complementing that stated in (5.12).

In the case of lines-of-curvature coordinate (cf Sections, 4, 5 and 8), the invariant tensor-form theory specializes to a modification of the Luré–Novozhilov formulation. This straightens the drawback, first indicated by Budiansky and Sanders (1963): "... the Luré–Novozhilov reduction performed in lines-of-curvature coordinates is not consistent with any general tensor representation of the modified stress variables". The consistency requires merely employing the appropriate strain and stress variables—the symmetric parts of the tensors $\rho_{\alpha\beta}$ and $n^{\alpha\beta}$ [determined in (4.8), (4.16), (5.8) and (5.14)]. Quantitatively, the modification concerns only the torsion and shear components (τ and S).

9. Intensely variable deformation. DMVK-theory

Most results on elastic buckling and postbuckling of shells are due to one specialized branch of the shell theory—that of 'quasi-shallow' shells. This theory achieves a striking simplification—all problems, linear and nonlinear, are solved with the aid of a system of merely two, reciprocally dual, equations.

The theory has been founded, nearly simultaneously, by Donnell (1933) and by Mushtary in the U.S.S.R. (Both treated the buckling of a circular cylinder in torsion.) A similar approach was developed in 1938 by Marguerre for shallow shells and in 1941 by Karman for plates. In 1944 Vlassov extended the theory to noncylinder shells and introduced the Airy function. However, the bending strains were still determined in terms of displacements, and this by the linear expressions with only the normal deflection retained. That is, the nonlinear problems were treated on the basis of an approximation of the linear relations (identical to those introduced in the first work on shells (Aron, 1987) and severely criticized ever since). The illegitimacy of this part of the Donnell theory for large rotations was certified in 1963 by Donnell himself in an unpublished lecture 'General shell displacement–strain relations'. But there was no alternative[1] till the work of Libai (1962) and Koiter (1966) gave the theory those main features it still has. Koiter (1966) termed it 'a theory of quasi-shallow shells or a theory for shells of

[1] Despite its illegitimate displacement-approach part, the theory did not encounter grave difficulties in applications. This 'skating on thin ice' (Koiter, 1966) was made possible by the concentration on a specific class of problems (cf Section 11)—those with a small wave-length deformation pattern (of buckling modes), which involves predominantly normal deflection and no large rotations.

small Gaussian curvature'. As discussed in the following, this theory is applicable to shallow shells not unconditionally. Therefore, it will be in the sequel referred to as Donnell–Mushtary–Vlassov–Koiter (DMVK) theory. Consider for this theory: (1) its standard formulation; (2) error estimates; and (3) physical hypothesis and applicability.

9.1.

The DMVK-theory starts in the relevant key work (Koiter, 1966) with *two assumptions*: (a) The maximum absolute wall-bending strain and middle-surface extension ($h\rho/2$ and η), in the shell region under consideration, are assumed to have comparable orders-of-magnitude in the sense

$$h/R \ll h\rho/\eta \ll \min(R/h, 1/h\rho). \tag{9.1}$$

On the basis of (9.1), Koiter (1966) introduces two simplifications of the general theory:

(i) The two equations of tangential-forces equilibrium are used without the terms depending, directly and through $n^{\alpha 3}$, on the moment $m^{\alpha\beta}$; the dual compatibility equations—without terms depending, directly or through λ_β, on the $E_{\alpha\beta}$.
(ii) The stress and strain tensors $n^{\alpha\beta}$, $\rho_{\alpha\beta}$ are set equal to their symmetric parts $(n^{\alpha\beta} + n^{\beta\alpha})/2$, $(\rho_{\alpha\beta} + \rho_{\beta\alpha})/2$.

This means dropping the terms with $E_{\alpha\beta}$ and λ_β in (4.7) and (4.4), as well as the terms with $m^{\alpha\beta}$ and $n^{\alpha 3}$ in (5.7) and (5.4). The equations become

$$\rho_{2\lambda;1} - \rho_{1\lambda;2} \cong 0, \quad \rho_{\alpha\beta} \cong \bar{\rho}_{\alpha\beta} = (\rho_{\alpha\beta} + \rho_{\beta\alpha})/2; \tag{9.2}$$

$$n^{1\lambda}_{;1} + n^{2\lambda}_{;2} + q^\lambda \cong 0, \quad n^{\alpha\beta} \cong \bar{n}^{\alpha\beta} = \left(n^{\alpha\beta} + n^{\beta\alpha}\right)/2. \tag{9.3}$$

(b) The Gaussian curvature K of the undeformed middle surface is assumed to be small compared with the square of the *minimum* interval of variation L:

$$KL^2 \ll 1. \tag{9.4}$$

This is considered to justify the interchange of the sequence of covariant surface differentiation, which makes the following expressions to a general solution of the simplified equations of equilibrium and compatibility (9.2) and (9.3):

$$\rho_{\alpha\beta} = W_{;\alpha\beta}, \quad n^{\alpha\beta} = \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}F_{;\lambda\mu} + P^{\alpha\beta}. \tag{9.5}$$

The $W(x^\alpha)$ and $F(x^\alpha)$ are a three times continuously differentiable curvature function and an Airy function; $P^{\alpha\beta}$ denotes a particular solution of the eqns (9.3).

The remaining equations of compatibility, (4.5) with (4.6), and of equilibrium, (5.5) with (5.6), render after inserting (9.5) the system of equations for F and W

$$\nabla^4 F + Eh\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}\left(b_{\alpha\beta} + W_{;\alpha\beta}/2\right)W_{;\lambda\mu} = 0, \tag{9.6}$$

$$D\nabla^4 W - \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}b'_{\alpha\beta}F_{;\lambda\mu} + q = 0. \tag{9.7}$$

Here $\nabla^4 = \nabla^2\nabla^2$ with ∇^2 denoting the two-dimensional Laplacian operator. The $P^{\alpha\beta}$-terms are not written out, for the sake of simplicity.

The above standard theory still leaves open three questions:

(i) Is the accuracy of the theory dependent only on the *maximum* values of the bending and membrane strain ($h\rho$, η), irrespective of their specific components?
(ii) Can the basic hypotheses be stated in terms which allow one to assess whether the theory is adequate for a problem, *before* its solution has been obtained?
(iii) Does the theory need to be defined by any physical criterion, *besides* that of shallow shape or curvature restriction?

9.2.

Consider the **accuracy** of the DMVK-theory, as determined by the error of its simplifications. This involves terms dropped in the equations of compatibility and equilibrium, first, to obtain the eqns (9.2) and (9.3), second, to satisfy these equations by the general solution (9.5).

The relative error δ_λ of the compatibility eqns (9.2) is determined by the relation of the terms of (4.4) dropped in (9.2) to one of the terms remaining there:

$$\delta_\lambda = \max\{|\varepsilon^{\alpha\beta} b^\lambda_\alpha \lambda_\beta| / |\varepsilon^{\alpha\beta}\varepsilon^{\lambda\delta}\rho_{\alpha\delta;\beta}|\}. \tag{9.8}$$

The dual to δ_λ error estimate Δ_λ of the equilibrium eqns (9.3) is determined by the relation of terms of (5.4) dropped in (9.3), to one of the terms retained there.

The estimate (9.8) depends on the assessment of the covariant derivatives. Such derivative with respect to *any* of the coordinates x^α is, following Koiter 1966, (p. 34), estimated by means of the *minimum* interval of variation L

$$|\rho_{\alpha\lambda;\beta\gamma}| \sim |\rho_{\alpha\lambda}|/L^2, \quad |b^\alpha_\gamma E_{\alpha\lambda;\beta}| \sim |E_{\alpha\lambda}|/(LR). \tag{9.9}$$

The factors b^α_γ are estimated here by the absolute maximum normal-section curvature $1/R$. Closer estimates can be obtained when the terms with b^2_1, b^1_2 may be dispensed with. This is the case, when the x^α-lines are the curvature lines, or follow these lines approximately. For such coordinates, the terms dropped in (4.4) and (5.4) to obtain (9.2) and (9.3) can be estimated in terms of physical components defined in (4.10) and (4.11) and $\gamma_{\alpha\alpha} = \varepsilon_\alpha$, $\gamma_{12} = \gamma_{21} = \gamma$

$$\delta_\lambda \sim \max |(E_{\alpha\beta}/\rho_{\alpha\beta}h)(h/R_\delta) \sim \max |(\gamma_{\alpha\beta}/\kappa_{\alpha\beta}h)(h/R_\delta)| \quad (\delta \neq \beta); \tag{9.10}$$

$$\Delta_\lambda \sim \max |(m^{\alpha\beta}/n^{\alpha\beta})/R_\beta| \sim \max |(\kappa_{\alpha\beta}h/12\gamma_{\alpha\beta})(h/R_\beta)|, \tag{9.11}$$

where R_δ and R_β denote the principal radii of curvature of the reference surface.

The estimates (9.10) and (9.11) indicate an answer to the question (i) of Section 9.1. However, these estimates depend on the values of the stress and strain resultants. The accuracy of the theory for a problem can be checked only *after* a solution for this problem has been evaluated. The way to avoid this difficulty and also the clarification of the above questions (ii) and (iii) is suggested by the general solution (9.5): all stress and strain variables can be expressed in terms of W and F. The $\rho_{\alpha\beta}$ and $n^{\alpha\beta}$ are given in terms of W and F directly by (9.5), and it is derivable:

$$n^{\alpha 3} = Da^{\alpha\lambda}(\nabla^2 W)_{;\lambda}, \quad Eh\varepsilon^{\alpha\beta}\lambda_\beta = -a^{\alpha\lambda}(\nabla^2 F)_{;\lambda}. \tag{9.12}$$

Further, it has to be accounted for, that the stress state can vary with the two coordinates x^α with

different intensities. To take this into consideration, a certain extension of the estimates (9.9) is indispensable. The covariant derivative with respect to x^α will be estimated by the formula (9.9), but with the L replaced by the interval L_α of variation with respect to the specific coordinate x^α:

$$|f_{\beta\delta;\alpha}| \sim |f_{\beta\delta}|a_\alpha/L_\alpha. \tag{9.13}$$

To assess the terms dispensed with in the DMVK-theory, it only remains to estimate the relation between F and W. This is done with the help of Eqs. (9.6) and (9.7) which provide relations between the $\nabla^4 F$ or $\nabla^4 W$ and the second term of the respective equation. These terms are, just as $\nabla^4 F$ and $\nabla^4 W$, invariants. They may be estimated for lines-of-curvature x^α. The relevant order-of-magnitude estimates, following from (9.6) and, respectively, (9.7) are:

$$|\nabla^4 F| \sim Eh|W|/|R_\alpha L_\beta^2|_{\min}, \quad D|\nabla^4 W| \sim |F|/|R_\alpha L_\beta^2|_{\min} \quad (\alpha \neq \beta). \tag{9.14}$$

Here L_β denote the values of the intervals of variation, defined as L_β by (9.13), when the x^β-lines coincide with the curvature lines; $|R_\alpha L_\beta^2|_{\min}$ is the smaller of the $|R_1 \mathrm{L}_2^2|$, $|R_2 \mathrm{L}_1^2|$.

The relations (9.12)–(9.14) lead to the estimates of the relative errors in the eqns (9.2) and (9.3). Remarkably, these errors have equal orders of magnitude:

$$\Delta_\lambda \sim \delta_\lambda \sim (L_1 L_2)^2/(R_\lambda |R_\alpha \mathrm{L}_\beta^2|_{\min}) \ (\alpha \neq \beta). \tag{9.15}$$

Further, errors of the DMVK-theory are caused by replacing the $\rho_{\alpha\beta}$ and $n^{\alpha\beta}$ by the symmetric parts $\bar{\rho}_{\alpha\beta}$ and $\bar{n}^{\alpha\beta}$ of the respective tensors. This amounts to an approximation of eqns (4.8) and (5.8). The estimates of the corresponding relative errors, determined similarly to (9.15), are:

$$|\rho_{\alpha\beta}/\bar{\rho}_{\alpha\beta} - 1| \sim |n^{\alpha\beta}/\bar{n}^{\alpha\beta} - 1| \sim |\max |L_\alpha| L^3/\left(R|R_\alpha L_\beta^2|_{\min}\right). \tag{9.16}$$

Finally, the inaccuracy, caused by the interchange of the sequence of covariant surface differentiation, turns out to be, in consequence of $L_\alpha \geqslant L$, somewhat higher than its estimate (9.4). However, this error, just as the one assessed in (9.16), are under those of the basic simplifications estimated in (9.15).

The estimates (9.15) and (9.16) do not depend on stress and strain variables and, thus, are in many cases serviceable for a problem without its solution having been evaluated (cf an example in Section 12).

9.3.

The **physical meaning** of the DMVK simplifications is displayed by the error estimates (9.15) and (9.16), in which L_α and L_α characterize the variability of the stress state, the R and R_α represent the variability of $\boldsymbol{n}$. The estimates show the DMVK-theory to be adequate for stress states which vary with both coordinates x^α much more intensely than the unit normal $\boldsymbol{n}$. This meaning of the theory is made more transparent by the simpler estimates which follow from (9.15) and (9.16) for the lines-of-curvature x^α. With the overall error of the DMVK-theory denoted Δ, this estimate is

$$\Delta \leqslant \Delta_\alpha \sim \delta_\alpha \sim L_\alpha^2/R_\alpha^2. \tag{9.17}$$

The estimates (9.17) define problems, for which the error of the DMVK-theory does not exceed that of the general thin-shell theory, by the conditions $L_\alpha^2/R_\alpha^2 \leqslant h/R$. Two examples illustrating the accuracy and the applicability limits of the DMVK-theory are discussed in Section 12. The answer to question (iii) of Section 9.1 is: the applicability of the theory of 'quasi-shallow' shells depends not on the shell shape alone, however shallow it may be.

The domain of the DMVK-theory, determined by the condition $\Delta \leqslant \max (L_\alpha/R_\alpha)^2 \leqslant h/R$, is plotted in Fig. 1, where the coordinates R_α/L_α represent the stress state and the geometry at a point.

10. Low variability stress state. Membrane theory

The resistance of a thin wall to bending is weak. Thin shells can withstand a large load only through the momentless, membrane, stress state. This kind of stress state is the optimum for shell structures designed for strength and stiffness.

The equilibrium of a shell element which is supported by the membrane resultants alone is described by the equations following from (5.4), (5.5) and (5.7) after dropping all their terms dependent on the moments $m^{\alpha\beta}$ and the transverse resultants $n^{\alpha 3}$:

$$n^{\alpha\beta}_{;\alpha} + q^\beta = 0, \quad n^{\alpha\beta} = n^{\beta\alpha} \tag{10.1}$$

$$b'_{\alpha\beta} n^{\alpha\beta} - q = 0. \tag{10.2}$$

The number of these equations (four) is equal to the number of the unknowns $n^{\alpha\beta}$. By appropriate boundary conditions, and by the compatibility equations approximately fulfilled, the resultants $n^{\alpha\beta}$ are statically determinate. Once the solution of the eqns (10.1) and (10.2) has been obtained, the wall bending can be evaluated and its influence estimated with the aid of the constitutive and compatibility equations. This will not be discussed further. The gaol will be a simple assessment of the applicability domain and the accuracy of the momentless theory. (The fundamental treatment of the membrane stress state is represented by the extensive asymptotic analysis—cf Goldenveizer, 1976 and Ciarlet and Lods, 1996.)

The momentless equations of tangential-force balance (10.1) are employed in the DMVK-theory, their accuracy has been estimated in (9.11). In terms of the components of stress, more significant for the membrane theory, the estimate (9.11) indicates for the error of the eqns (10.1) the order:

$$\Delta_\alpha \sim \max |\left(\sigma^{\alpha\beta}_m/\sigma^{\alpha\beta}_n\right) h/6R_\beta|. \tag{10.3}$$

Where $\sigma^{\alpha\beta}_m = |m^{\alpha\beta}|6/h^2$ denotes the maximum through the wall thickness stress resulting in the bending and torsional moments: $\sigma^{\alpha\beta}_n = |n^{\alpha\beta}|/h$ represents the stress component resulting in $n^{\alpha\beta}$; $1/R_\beta$ is the principal curvature.

The relative error of the eqn (10.2) is estimated by the relation of the dropped $n^{\alpha 3}$-term of (5.5) to the main term of those, $n^{\alpha\beta} b'_{\alpha\beta}$, retained in (10.2). With the estimate of the quantity $n^{\alpha\beta} b_{\alpha\beta}$, similar to that used in (9.14) the error of (10.2) has the magnitude (no summation here):

$$\Delta \sim \max|n^{\alpha 3}_{;\alpha}|/\max|n^\alpha_\beta/R_\alpha|. \tag{10.4}$$

The expression of $n^{\alpha 3}$ in terms of $m^{\lambda\alpha}$ follows from (5.6). The assessment (9.13) of the covariant derivatives (admittedly, in this case merely a rough estimate) leads to the estimates of the derivatives $|n^{\alpha 3}_{;\alpha}| \sim |m^{\beta\alpha}|/L_\alpha L_\beta$. The formula (10.4) becomes

$$\Delta \sim \max|m^{\beta\alpha}/L_\alpha L_\beta|/\max|n^\alpha_\beta/R_\alpha|.$$

This estimate is now to be approximated by a simpler and more conservative one. This is done with $L_\alpha L_\beta$ and R_α replaced by the smaller quantities L^2 and R, as well as with the $m^{\alpha\beta}$ and n^α_β represented by the corresponding maximum absolute bending stress $\sigma^{\alpha\beta}_m$ and, respectively, the membrane stress $\sigma^{\alpha\beta}_n$. The

estimate (10.4) of the error of the membrane eqn (10.2) becomes

$$\Delta \sim \max\left(\sigma_m^{\alpha\beta}/\sigma_n^{\alpha\beta}\right) hR/6L^2. \tag{10.5}$$

The estimates (10.3) and (10.5) depend on the relation of the bending and membrane stresses in a problem under consideration. That is, the estimates are determined in terms of quantities known only after the solution of the problem has been obtained. In the membrane theory there is no possibility to eliminate the stress resultants from the error estimates. (In the DMVK-theory, Section 9, this has been made feasible by the general solution F, W.)

However, the purpose, the very meaning of the membrane theory, involves a restriction on the value of just this relation $\sigma_m^{\alpha\beta}/\sigma_n^{\alpha\beta}$. Indeed, although the bending stress $\sigma_m^{\alpha\beta}$ cannot be totally avoided in real structures under realistic loads, this theory has a practical sense only if the order of magnitude of the bending stress $\sigma_m^{\alpha\beta}$ does not, at any rate, exceed that of the membrane stresses $\sigma_n^{\alpha\beta}$. That is, the membrane theory is intended for the cases when

$$\sigma_m^{\alpha\beta}/\sigma_n^{\alpha\beta} < 0(1). \tag{10.6}$$

The magnitude of Δ is, according to (10.5) and (10.6), higher than that of Δ_α from (10.3). Thus, the Δ assesses the overall error of the membrane theory system (10.1) and (10.2). With this, the error, determined by (10.3)–(10.5), does not exceed in its magnitude order that of the general theory when $\Delta \leqslant h/R$. Allowing for the bending stresses, as their upper limit, that in (10.6), the accuracy condition of the membrane theory is

$$\Delta \sim hR/6L^2 \leqslant h/R. \tag{10.7}$$

This condition amounts to $R^2/L^2 \leqslant 6$. It delimits the domain of the membrane theory on hand of the relation of the intensities of variation: of $\boldsymbol{n}$, represented by the normal-section curvature $1/R = \max|1/R_\alpha|$, and of the stress state, represented by $1/L = \max(1/L_\alpha)$. The membrane theory is adequate for the stress states varying not much stronger than the unit normal vector $\boldsymbol{n}$. The applicability domain of the membrane theory (MT) is plotted in Fig. 1.

11. Large displacements by small strain. Flexible-shell theory

Large displacements and rotations have for decades been the prime target of the elastic-shell theory. The axisymmetrical and other one-dimensional nonlinear problems have their effective specialized treatment (cf references in Section 1). However, the efforts to solve the nonlinear shell problem for arbitrary geometry and arbitrary stress state by frontal assault of the general theory have not produced concepts expedient for applications. The displacement approach, characteristic for these attempts, was not helpful.

The nonlinear problems of shell structures are of *two classes*. The *first* concerns shells designed for strength and *stiffness*. These problems are characterized by: (a) a membrane stress as an optimum; (b) the only states which may involve substantial nonlinearity before collapse are buckling and postbuckling; (c) rotations remain small. The nonlinear analysis has been for a long-time concentrated on problems of this class. This has been with rare clarity indicated by Koiter 1966, (pp. 38 and 44): "...(nearly) inextensional bending,... excluded in well designed shell structures". "Comparatively small wavelengths of the deformation pattern on the middle surface are indeed a common feature of most nonlinear shell problems,...". Another key work (Goldenveizer, 1976, p. 103) tells: "... the basic stress state is momentless (if it does not degenerate)".

Problems of this class are adequately described by the DMVK-theory (Section 9) or, in the cases of elongated buckles, when $L_1 \gg L_2$, by the flexible-shell theory discussed in the sequel.

The *second class* of problems concerns shells designed for elastic *deformability*, for large displacements and rotations—*flexible shells*. To achieve flexibility, the stress state has to be of the wall-bending kind. The membrane deformation cannot contribute significant displacements. (To recall: by a relative extension ε, a length l extends merely by εl.) Moreover, not any kind of wall-bending leads to deformability. In particular, flexibility cannot be accomplished by the deformation of small-wavelengths pattern—the rotation is an integral of the bending strain.

Consider for the flexible-shell theory (FST): 1. The basic hypothesis. 2. Compatibility and equilibrium. 3. Elastic energy, constitutive equations and the resolving system. 4. Extended FST. 5. Applicability domain.

11.1.

The hypothesis, sufficient to reduce the general theory to the FST, consists of one single assumption reflecting the feature characteristic for all flexible shells despite their diversity (cf Axelrad, 1987, 1992): the local shape of the shell and its basic stress state vary with one coordinate x^α less intensively than with the other.

In terms of the relation f of the intensities of variation of the local shape and of the stress state $(1/R_\alpha, 1/L_\alpha)$ the hypothesis reads:

$$|\boldsymbol{a}_{j,1}/a_1| \leqslant f|\boldsymbol{a}_{j,2}/a_2|, \quad |a_{j,1}/a_1| \leqslant f^2|a_{j,2}/a_2|; \tag{11.1}$$

$$f = L_2/L_1, \quad f^2 \ll 1, \quad |R_2/R_1| \sim f, \quad |L_\alpha/R_\alpha \leqslant 0(1). \tag{11.2}$$

As defined above: $j = 1, 2, 3$; $a_j = |\boldsymbol{a}_j|, \boldsymbol{a}_3 = \boldsymbol{n}$.

The basic assumption shows the FST to be not indifferent to the choice of surface coordinates x^α. The coordinate lines, for which the ratio R_2/R_1 of the normal-section curvatures is minimum, are preferable for the formulation of the FST. These x^α-lines—the curvature lines of the surface—proved advantageous for the analysis of all flexible-shell structures, known to the author. Such coordinates x^α, with $a_{12}, b_{12} = 0$, are used for the FST in the sequel. The resulting formulation of the theory can by the change of coordinates be cast in the invariant tensor form (discussed by Axelrad and Emmerling, 1987).

11.2.

Consider the **simplifications** in the compatibility and equilibrium equations, to be inferred from the FST hypothesis, and the relevant *errors*.

The starting point provide eqns (4.2) and (5.2), which indicate for their main terms nearly equal orders of magnitude: $|\boldsymbol{K}_{1,2}| \sim |\boldsymbol{K}_{2,1}|, |\boldsymbol{T}^1_{,1}| \sim |\boldsymbol{T}^2_{,2}|$. These relations, together with the hypothesis (11.1) and (11.2) and the estimates (8.2) for the derivatives, lead to $|\boldsymbol{K}_1| \sim f|\boldsymbol{K}_2|, |\boldsymbol{T}^2| \sim f|\boldsymbol{T}^1|$, which, decomposed in physical components, defined in (3.4), (4.11) and (5.11), yield:

$$|\kappa_{1\alpha}\boldsymbol{n}^* \times \boldsymbol{t}^\alpha + \kappa_{13}\boldsymbol{n}^*| \sim f|\kappa_{2\alpha}\boldsymbol{n}^* \times \boldsymbol{t}^\alpha + \kappa_{23}\boldsymbol{n}^*|, \quad |N^{2j}\boldsymbol{t}_j| \sim f|N^{1j}\boldsymbol{t}_j|. \tag{11.3}$$

As follows from the eqns (4.12), (4.16), (5.10) and (5.12): $\kappa_{12} \sim \kappa_{21} \sim \tau$, $N^{21} \sim N^{12} \sim S$.

Further, the $\boldsymbol{n}^*$ components in (11.3) ($\kappa_{\alpha 3}$ and $N^{\alpha 3}$) are, at most, of the order of magnitude of the tangential components. With this, the eqns (11.3) render the order-of-magnitude relations

$$\max|\kappa_{1\alpha}, \tau| \sim f|\kappa_{22}|, \quad \max|N^{2\alpha}, S| \sim f|N^{11}|. \tag{11.4}$$

With the variables expressed by the elasticity relations (8.15), the estimates (11.4) indicate further four resultants which are of secondary magnitude, and later turn out to be dispensable in the FST:

$$\max|\varepsilon_2, \gamma G/E| \sim |\varepsilon_1|, \quad \max|M_1, HE/2G| \sim f|M_2|. \tag{11.5}$$

Consider the relative magnitude of the terms with the $\varepsilon_2, \gamma, M_1$ and H in the equations of compatibility and equilibrium.

The eqns (4.13)–(4.16) can be written in the form:

$$\kappa_{2,1} = \left(a_1^2\tau\right)_{,2}/a_1 + \kappa_{2,\,1}\delta_1,$$

$$\tau_{,1} = (a_1k_1)_{,2}/a_2 + \kappa_2 a_1/R_{13} - (a_1\varepsilon_1)_{,2}/a_2R_2' + \tau_{,1}\delta_2; \tag{11.6}$$

$$\kappa_1 = -\kappa_2 R_2'/R_1 - (a_1\varepsilon_1)_{,22}R_2'/a_1a_{22} + \tau^2 R_2' + \kappa_1\delta_3. \tag{11.7}$$

The $\delta_j = \delta_j(\gamma, \varepsilon_2)$ denote the sum of all terms of the corresponding equation, which include γ and ε_2. With the estimates (11.5) and (11.2), the relative errors of dropping in (11.6) and (11.7) the terms with γ and ε_2 turn out to have the orders of magnitude:

$$\delta(\gamma, \varepsilon_2) = |\delta_3| \sim |(\varepsilon_1/\kappa_2)(E/G + L_1/R_{13})R_2/L_1L_2|, \quad |\delta_\alpha| < \delta. \tag{11.8}$$

The relation ε_1/κ_2 remains to be estimated.

The equations of equilibrium dual to (11.6) and (11.7) follow from these according to the relations (5.12) and (5.13). [The τ^2R_2' of (11.7) is replaced by two terms indicated by the duality (5.12): the first, $S\tau R_2' = SR_2'/R_{21}'$ is the dual to $\tau^2R_2' = \tau R_2'/R_{21}'$; the second, $S\tau R_2'$—dual of the term $\tau R_2'/R_{21}$, that happens to be zero in (11.7) because of $1/R_{21} = 0$.] All terms of the equilibrium equations containing the variables H and M_1 are represented by expressions $\Delta_j(H, M_1)$ dual to the $\delta_j(\gamma, \varepsilon_2)$ of (11.6) and (11.7). It can be found similarly to (11.8)—with the aid of the estimates (11.5) and (11.2)—that the relative errors of dropping the H-, M_1-terms have the magnitude

$$\Delta(H, M_1) = |\Delta_3| \sim |(M_2/N_1)(4G/E + L_1/R_{13})R_2/L_1L_2|, \quad |\Delta_\alpha| < \Delta. \tag{11.9}$$

The quantity M_2/N_1 remains to be estimated.

Terms which are in the FST small, include, besides those represented by $\delta_j(\gamma, \varepsilon_2), \Delta_j(H, M_1)$, the nonlinear terms: τ^2R_2' in eqn (11.7) and $2\tau SR_2'$ in the equation of equilibrium dual to (11.7). According to (11.4), τ and S are of the order of f in relation to κ_2 and, respectively, to N_1. In relation to the left side of the corresponding equation, i.e., to κ_1 in (11.7) or to N_2 in the dual equilibrium equation, these nonlinear terms are in the thin-shell theory negligible. Indeed, with $\kappa_2 h \sim \eta$, there is:

$$|\tau^2R_2'/\kappa_1| \sim f|\kappa_2R_2'| \sim f\eta R_2'/h, \quad |\tau SR_2'/N_2| \sim f|\kappa_2R_2'| \sim f\eta R_2'/h. \tag{11.10}$$

When the terms with γ, ε_2 and those estimated in (11.10), are dropped, the compatibility eqns (11.6) and (11.7) can be reduced to one equation with two variables κ_2, ε_1; the dual set of equilibrium equations without H-, M_1-terms can be reduced to an equation for N_1 and M_2. The two equations read:

$$\mathrm{V}[a_1\kappa_2] + \mathrm{W}[a_1\varepsilon_1] = 0,$$

$$V'[a_1N_1] - \mathrm{W}[a_1M_2] = -a_1^2q_{1,1} + \partial_2\left(a_1^3q_2\right) + \partial_2a_1^2\partial_2\left(a_1R_2'q\right); \tag{11.11}$$

where it is denoted

$$\mathrm{V}[\,] = [\,]_{,11} - \partial_2\big([\,]\, a_1^2/R_{13}\big) + \partial_2 a_1^2 \partial_2\big([\,]\, R_2'/R_1\big), \quad \partial_2[\,] = \partial[\,]/a_2\partial x^2,$$

$$W[\,] = \partial_2\big(a_1^2/R_2'\big)\partial_2[\,] + \partial_2 a_1^2 \partial_2 R_2' \partial_2 \partial_2[\,].$$

The operator $\mathrm{V}'[\,]$ is defined as $\mathrm{V}[\,]$ with R_{13} and R_1 replaced, respectively, by R_{13}' and R_1'. This reflects the duality of the nonlinear equations indicated in (5.13).

For cylinder shell, in linear approximation, $\mathrm{V}[\,] = [\,]_{,11}$, $R_2' = R_2$ and, with the elasticity relations $\varepsilon_1 = N_1/Eh$, $M_2 = D\kappa_2$ discussed in the following, the eqns (11.11) reduce to those of Vlassov (cf Novozhilov, 1970).

The evaluation of the FST errors can now be completed. For this purpose, the necessary estimates of ε_1/κ_2 and M_2/N_1 follow with (11.5), (11.1) and (11.2) from (11.11). Significantly, the two quantities are of the same magnitude order:

$$|\varepsilon_1/\kappa_2| \sim |M_2/N_1| \sim L_2^2/|R_1|. \tag{11.12}$$

With this, the relative errors (11.8) and (11.9) have the magnitudes:

$$\delta(\gamma, \varepsilon_2) \sim f^2 E/G + f^2 L_1/|R_{13}|, \quad \Delta \sim f^2 4E/G + f^2 L_1/|R_{13}|. \tag{11.13}$$

This shows terms with γ, ε_2 in the equations of compatibility and terms with H, M_1 in the equations of equilibrium to have the relative magnitude of f^2E/G. The hypothesis and its expressions (11.2) infer dropping these terms.

11.3.

The **elastic-energy** density V, must satisfy the conditions of consistency with the equations of compatibility and equilibrium (Axelrad and Emmerling, 1990). In the present case, when in these equations the terms with γ and ε_2 and, respectively, H and M_1 have been dropped, the consistency conditions are

$$\partial V/\partial S = \gamma = 0, \quad \partial V/\partial \tau = 2H = 0, \ \partial V/\partial N_2 = \varepsilon_2 = 0, \quad \partial V/\partial \kappa_1 = M_1 = 0. \tag{11.14}$$

With (11.4), there follows from (8.14) the energy density and the full set of two elasticity equations:

$$2V = N_1^2/Eh + D\kappa_2^2, \tag{11.15}$$

$$\varepsilon_1 = \partial V/\partial N_1 = N_1/Eh, \quad M_2 = \partial V/\partial \kappa_2 = D\kappa_2. \tag{11.16}$$

The equations of elasticity (11.16) require a cautionary remark. They differ from the general Eq. (8.15) by the absence of terms $-\nu N_2/Eh$ and $\nu D\kappa_1$. However, some works (treating axisymmetric and the related Saint-Venant problems) neglect not the terms with N_2 and κ_1, dropped in Eq. (11.16) as demanded by the consistency, but those with ε_2 and M_1, which are neglected in the compatibility and equilibrium. This produces equations $\varepsilon_1 = (1-\nu^2)N_1/Eh$ and $M_2 = (1-\nu^2)D\kappa_2$ differing from the correct relations (11.16) by what one of these publications terms "the factor $1-\nu^2$ which ghosts in the literature…". Regrettably, 'the literature' includes reputable work.

The two eqns (11.11) and the elasticity eqns (11.16) constitute a closed system which determines the N_1 and κ_2. All stress and strain resultants and the energy density, are expressed in terms of the N_1 and

κ_2. The variables N_1 and κ_2 take the role of a *general solution*—the role similar to that of the Airy function F and curvature function W in the DMVK-theory.

Despite the simplifications, particularly far reaching in the energy expression (11.15) and in the elasticity Eqs. (11.16), inside a wide domain of problems the FST is adequate in accuracy to the general shell theory. This is displayed by all known (flexible) shells designed for small-strain large displacements, i.e. displacements of the magnitude order of the overall dimensions of the shell.

11.4.

The **FST can be extended** to encompass the stress states with $f = L_2/L_1$ as large as 0(1) and/or with the structural anisotropy of $E/G \gg 1$, i.e., with weak stiffness with respect to shear or to bending. Such cases are out of reach of the above formulation of FST—the estimates (11.13) give the overall error of the theory $\Delta \sim f^2 4E/G \sim 1$.

The indicated extension of the FST is achieved by *retaining the shear γ and the torsional moment H*. This involves merely moderate complications—the γ and H are simply expressed by the elasticity relations in terms of S and τ, variables indispensable in the FST.

The effect of retaining γ and H on the accuracy of the FST is clearly displayed by the estimates (11.13). The error caused by dropping the γ- and H-terms, is reflected in (11.13) by the terms f^2E/G and $f^2 4G/E$. The error of the extended theory, which drops ε_2 and M_1 but not the γ- and H-terms, has according to (11.13) the estimate:

$$\delta \sim \Delta \sim f^2 L_1/|R_{13}|. \tag{11.17}$$

For many flexible-shell problems, e.g. for those of tube bends and bellows, L_2 is small compared to $|R_{13}| \sim |R_1|$, which makes the error (11.17) small even by $f \sim 1$. Indeed, $L_2/L_1 = f \sim 1$ means $L_1 \sim L_2$. This makes $\delta \sim f^2 L_1/|R_{13}| \sim L_2/|R_{13}| \ll 1$.

For the theory retaining γ and H, the four consistency conditions employed in (11.14) render:

$$2V = N_1^2/Eh + D\kappa_2^2 + S^2/Gh + \tau^2 Gh^3/6, \tag{11.18}$$

$$\varepsilon_1 = N_1/Eh, \quad M_2 = D\kappa_2, \quad \gamma = S/Gh, \; H = \tau Gh^3/6. \tag{11.19}$$

The γ- and H-terms preclude the reduction to the system (11.11). The resolving system can consist of the compatibility eqns (11.6) and (11.7) and the two dual equations of equilibrium. This system determines the variables κ_2, τ and N_1, S, which describe the deformation and the stress of the sections $x^1 =$const or of the corresponding shell boundaries.

The extended FST, goes back to the analysis of short-radius tube bends, which exemplifies the properties of this theory and the FST boundary conditions. This formulation of FST, and its alternative—one having as a one-dimensional limit case the Reissner axisymmetric equations—have also been useful in the nonlinear and nonaxisymmetric analysis of bellows (Axelrad, 1980, 1987, 1992).

Retaining the shear strain γ was originally proposed by Schnell (1957), for cylinder shells.

11.5.

The **physical definition** and the applicability **domain** of the FST are the further inferences of its hypothesis. As discussed above, the theory dispenses with ε_2 in the equations of compatibility and with M_1 in equilibrium. The same is true for the boundary conditions, which result with the FST hypothesis from the variational principles. At an edge $x^1 =$ const, the theory sets only two boundary conditions,

those of the membrane theory. At the same time, for the sections $x^2 = \text{const}$, FST retains in its equations and boundary conditions, all resultants of the general theory.

Thus, the FST represents adequately the stress states which are free of substantial bending in the sections $x^1 = \text{const}$, but may have it in the sections $x^2 = \text{const}$. This two-fold approach is summed up by the term *semi-momentless* theory (the term applied by Novozhilov, 1970, for the linear theory of cylinder shells, developed in 1930 by Vlassov, Pasternak, a.o.).

What the FST excludes, by dispensing with ε_2- and M_1-terms, is the part of the stress state which varies with x^1 intensively. When caused by forces acting at the line $x^1 = \text{const}$, in particular, on an edge, this part of deformation fades out inside a short distance $\sim\sqrt{hR}$ from the zone of loading. It is known as the edge effect. The boundary conditions at an edge $x^1 = \text{const}$, excluded by the FST, are the principal two of the (four) conditions of the edge effect. If required, the edge-effect part of the stress state can be superimposed on the part determined by the flexible shell theory.

The FST focuses on the 'main part' of the stress state, one extending far over the shell from the zone of loading or from an edge. (The partition of the stress-state into the main part and the edge-effect has been intensively investigated for cylinder shells by Goldenveizer, 1970).

Advantages of the FST are given by simplified equations of lower order in x^1, by specialized variational formulations and the physical transparency resulting from the hypothesis, and last, but not least, FST facilitates numerical solutions—the intensively variable in x^1 component of the stress state being excluded, it need not be eliminated numerically. This reduces the number of the required finite elements or integration intervals by factor h/R.

The attributes of the FST are also helpful by *physical* nonlinearities. The equilibrium and compatibility equations of the FST provide a closed system with merely two relevant constitutive equations. The two serve the roles analogous to those of the eqns (11.16) $\varepsilon_1 = N_1/Eh,\ M_2 = D\kappa_2$. (A case known to the author: a physically nonlinear FST solution yields the external collapse pressure for a bellows in a state of large elastic–plastic axisymmetric deformation. The pressure is rendered by a PC-program. It is confirmed by experiments.)

The flexible-shell theory is adequate in accuracy to the general theory when the relative magnitude of its overall error—the main term of Eq. (11.13)—is under h/R. This defines the FST domain (Fig. 1) by the condition

$$\delta(\gamma, \varepsilon_2) \sim \Delta(H, M_1) \sim (L_2/L_1)^2 4E/G \leqslant h/R. \tag{11.20}$$

12. Concluding remarks

While the applicability of the membrane and DMVK theories sets for both intervals of variation L_α the same bounds, the accuracy condition (11.20) of the FST states for the L_1 and L_2 different, indeed, opposite requirements. As should be expected recalling the hypotheses of the three specialized theories, many of the problems adequately described by one of them are governed by neither of the others. This is displayed by their applicability domains sketched in Fig. 1.

Compare the accuracy and applicability of the DMVK-theory and of the FST on examples, which treat the buckling of two circular cylinder shells under radial pressure.

By appropriate conditions on the edges supported in their planes, the buckling mode (cf, e.g., Axelrad, 1983, p. 191) consists of $n = 2, 3, \ldots$ circumferential waves extending over the entire length l of a cylinder shell with radius R and wall thickness h:

$$\kappa_2 = C \sin(x_1/L_1) \sin(x_2/L_2), \quad C = \text{const}, \quad L_1 = l/\pi, \quad L_2 = R/n, \quad f = \pi R/nl. \tag{12.1}$$

The DMVK-theory gives the critical pressure p for this mode (cf the reference above):

$$pR^3/D = n^2\left(1+f^2\right) + 12(1-\nu^2)f^4(R/h)^2/\left[n^2\left(1+f^2\right)\right]. \tag{12.2}$$

The FST renders for this problem

$$pR^3/D = n^2 - 1 + 12(1-\nu^2)f^4(R/h)^2/(n^2-1). \tag{12.3}$$

The result (12.3) differs from (12.2) in two points which display the two divergent basic features of the respective specialized theories. The replacement of n^2 in (12.2) by n^2-1 in (12.3) reflects the different treatment of the *tangential* balance of forces and strains. Namely, the terms with n^{23} and λ_1 are dropped in the DMVK-theory but retained in the FST. The second difference—the replacement of $1+f^2$ in (12.2) by 1 in (12.3)—reflects the unequal treatment of the balance of forces and strains in the *normal* direction. The terms $n^{13}_{;1}$ and $\lambda_{2;1}$ of eqns (5.5) and (4.5) are retained in the DMVK-theory but dropped in the FST.

We compare the numerical results of the two theories for (i) a short cylinder, buckling in a stiff-shell manner, and (ii) for a long cylinder, with the buckling pattern of the flexible-shell class.

(i) For the short shell, with $l = R$ by $R = 50h$, the DMVK-solution (12.2) determines $n = 7$ and the buckling pressure $p = 86.1D/R^3$. The accuracy of this result is adequate to the general shell theory: with $L_2 = R/n$ from (12.1), the error estimate (9.17) gives $\Delta \sim (L_2/R)^2 = 1/49 \approx h/R$.

For the same cylinder, the FST-solution (12.3) yields $n = 7$ and $p = 71.5D/R^3$, that is 17% less than $p = 86.1D/R^3$ of the DMVK-theory. In this case $f^2 = (\pi R/nl)^2 = 0.201$, which shows the problem to be outside the adequacy domain of the FST (which is $f^2 \ll 1$). The estimate (11.13) gives the FST error $\Delta \sim 0.201\,4E/G \approx 2.0$—the actual difference found above is merely 17%. This case shows the applicability of the FST to be wider than the domain estimated by (11.13).

(ii) For the long shell, $l = 20R$, $R = 50h$, the FST-solution (12.3) yields $n = 2$ and $p = 3.346D/R^3$. With $f = \pi R/nl = 0.07855$, the error estimate (11.13) is: $\Delta \sim 0.006E/G \sim h/R$. This indicates the FST result to involve no additional error, compared to the general theory.

For the same cylinder the DMVK-theory (12.2) gives $n = 2$ and $p = 4.305D/R^3$, which is 28% above the correct FST result $p = 3.346D/R$. The error estimate (9.17) of the DMVK-theory gives $\Delta \leqslant 1/n^2 = 0.25$, which confirms the problem to be outside the DMVK domain.

In both cases in which the error is substantial, it is less than the estimate—the estimates rate the *partial* errors, caused by approximations in the compatibility *or* in equilibrium. These errors are mostly of the same magnitude order and may compensate each other. It is an advantage of the mixed, intrinsic formulation.

References

Alumae, N.A., 1956. On the form of basic relations of the nonlinear shell theory (in Russian). Prikl. Mat. Mekh. (PMM) 20, 136–139.

Aron, H., 1874. Das Gleichgewicht und die Bewegung einer unendlich dünnen, beliebig gekrümmten elastischen Schale. J. f. reine und angew. Mathem 78, 136–173.

Axelrad, E.L., 1960. Equations of deformations for shells of revolution and for bending of thin-walled tubes subjected to large elastic displacements. Am. Rocket Soc. J. (translated) 32, 1147–1151.

Axelrad, E.L., 1962. Flexure and stability of thin-walled tubes under hydrostatic pressure (in Russian). Izvestiya Akademii Nauk SSSR, Mekhanika 4, 133–139.

Axelrad, E.L., 1965. Refinement of buckling-load analysis for tube flexure by way of considering precritical deformation (in Russian). Izvestiya Akademii Nauk SSSR, Mekhanika 4, 133–139.
Axelrad, E.L., 1967. On different definitions of measures of shell-curvature change and on compatibility equations. Mech. of Solids (translated) 2, 105–108.
Axelrad, E.L., 1980. Flexible shells. In: Rimrott, F.P.J., Tabarrok, B. (Eds.), XV Intern. Congr. of Theoret. and Appl. Mech. North-Holland, Amsterdam, pp. 45–56.
Axelrad, E.L., 1981. On vector description of arbitrary deformation of shells. Int. J. Solids and Struct 17, 301–304.
Axelrad, E.L., 1983. Schalentheorie. Teubner, Stuttgart.
Axelrad, E.L., 1987. Theory of Flexible Shells. North-Holland, Amsterdam.
Axelrad, E.L., 1992. Flexible shell theory and analysis of tubes and bellows. Journal of the Brazilian Society of Mechanical Sciences 14 (4), 369–386.
Axelrad, E.L., Emmerling, F.A., 1987. Intrinsic shell theory relevant for realizable large displacements. Int. J. Non-Linear Mechanics 22, 139–150.
Axelrad, E.L., Emmerling, F.A., 1988. Vector and tensor form of intrinsic shell-theory relations. Int. J. Non-Linear Mech 23, 9–23.
Axelrad, E.L., Emmerling, F.A., 1990. On variational principles and consistency of elasticity relations of thin shells. Int. J. Non-Linear Mech 25, 27–44.
Budiansky, B., Sanders, J. L., 1964. On the "best" first-order linear shell theory. Progress in Applied Mechanics, The Prager Anniversary Volume. New York, pp. 129–140.
Calladine, C.R., 1988. The theory of thin shell structures 1888–1988. The Institution of Mechanical Eng. Proceedings 202 (42).
Ciarlet, P.G., Lods, V., 1996. Asymptotic analysis of linearly elastic shells. Arch. Rational Mech. Analysis 136 (2), 119–200.
Danielson, D.A., 1992. Vectors and Tensors in Engineering and Physics. Addison-Wesley, Redwood City.
Donnell, L. H., 1933. Stability of thin-walled tubes under torsion. NACA Report 479.
Emmerling, F.A., 1982. Nonlinear bending of elliptic tubes under normal pressure. Ing.-Arch 52, 1–16.
Goldenveizer, A.L., 1976. Theory of Elastic Thin Shells (in Russian). Nauka, Moscow.
Ibrahimbegovic, A., 1997. Stress resultants and geometrically exact shell theory for finite rotations and its finite element implementation. Appl. Mech. Rev 50 (4).
Koiter, W.T., 1960. A consistent first approximation in the general theory of thin elastic shells. In: Proc. IUTAM Symp. Delft. North-Holland, Amsterdam.
Koiter, W.T., 1966. On the nonlinear theory of thin elastic shells. Proc. Kon. Neth. Akad. Wet B 69, 1–54.
Koiter, W.T., Simmonds, J.G., 1973. Foundations of shell theory. In: Becker, E., Mikhailov, G.K. (Eds.), Proc. 13th Congress on Theoret. and Appl. Mech., Springer, pp. 150–176.
Libai, A., 1962. On the nonlinear elastokinetics of shells and beams. J. Aerosp. Sci 29, 1190–1195.
Libai, A., Simmonds, J.G., 1983. Nonlinear Elastic Shell Theory. Academic Press, San Diego.
Libai, A., Simmonds, J.G., 1988. The Non-Linear Theory of Elastic Shells. One Spatial Dimension. Academic Press, Boston, San Diego.
Libai, A., Simmonds, J.G., 1998. The Nonlinear Theory of Elastic Shells, 2nd ed. Cambridge University Press, Cambridge.
Luré, A.I., 1940. General theory of thin elastic shells (in Russian). Prikl. Mat. Mekh. (PMM) 5, 7–34.
Luré, A.I., 1970. Theory of Elasticity (in Russian). Nauka, Moscow.
Novozhilov, V.V., 1970. Thin Shell Theory. Noordhoff, Groningen.
Reissner, E., 1981. On finite pure bending of curved tubes. Int. J. Solids and Struct 17, 839–844.
Reissner, E., 1996. Selected Works in Applied Mechanics and Mathematics. Jones and Bartlett, Boston.
Reissner, H., 1912. Spannungen in Kugelschalen. Festschrift H. Müller-Breslau, Leipzig, pp. 181–193.
Sanders, J.L., 1963. Nonlinear theories for thin shells. Q. Appl. Math 21, 21–36.
Schiek, B., Pietraskiewicz, W., Stumpf, H., 1992. Theory and numerical analysis of shells undergoing large elastic strains. Int. J. Solids Struct 29, 689–709.
Schnell, W., 1957. Zur Krafteinleitung in die versteifte Zylinderschale. Zeitschr. für Flugwiss. 3, 4.
Simmonds, J.G., 1979. Special cases of non-linear shell equations. In: Besseling, J.F., Heiden, A.M.A. (Eds.), Trends in Solids Mechanics. Sijthoff and Nordhoff, Leiden.
Simmonds, J.G., 1984. The non-linear thermodynamical theory of shells. In: Axelrad, E.L., Emmerling, F.A. (Eds.), Flexible Shells. Springer, Berlin.
Simmonds, J. G., 1997. Some comments on the status of the shell theory at the end of the 20th century: complaints and correctives. Amer. Inst. of Aer. and Astron.
Simmonds, J.G., Danielson, D.A., 1970. Nonlinear shell theory with a finite-rotation vector. Proc. Koninkl. Nederlandse Akad. Van Wetensch B 73, 460–478.
Weatherburn, C.E., 1925. On small deformations of surfaces and of thin elastic shells. Quart. J. Math 50, 292–296.